Generalized Permutations and Combinations

3/19/12

Note Title

Counting with n distinct objects

Type	Repetitions Allowed	Formula
r-perm.	NO	$P(n,r) = \frac{n!}{(n-r)!}$
r-combinations	NO	$C(n,r) = \frac{n!}{(n-r)!\,r!}$
r-perm.	yes	n^r
r-comb.	yes	

$$P(n,n) = n!$$

$$C(n,n) = \frac{P(n,n)}{n!} = 1$$

Combinations with repetition

Example:
Three schools -- A, B and C -- are competing for a grand prize in a science fair competition. There are two judges. Each judge, anonymously, recommends one of the two schools. Calculate the number of ways the judges recommend the schools.

* - recommendation

A B C

** | |

* | * |

* | | *

| ** |

| * | *

| | **

each row is an arrangement of 2 stars & 2 stripes

2 judges → 2 stars; # schools − 1 → 2 stripes

4 to arrange

$\binom{4}{2}$

The same as the number of 4-bit strings with exactly two 1s.
bit positions = # schools -1 + # of judges

n - schools
r - judges

$\binom{n-1+r}{r}$

Theorem: C(n-1+r,r) is the number of r-combinations of n elements with repetition.

Section 6.5, Problem 9

A bagel shop has eight different types of bagels. How many ways are there to choose

(a) Six bagels?

$$\binom{\overset{n}{8}-1+\overset{r}{6}}{6} = \binom{13}{6}$$

bagel 1 2 3 4 5 6 7 8

$** \mid\ \mid ** \mid\ \mid * \mid * \mid\ \mid$

One possible selection of bagels

(b) a dozen bagels? $r=12$, $n=8$

$$\binom{8-1+12}{12} = \binom{19}{12} = \binom{19}{7}$$

$\because \binom{a}{b} = \binom{a}{a-b}$ $\quad 0 \le b \le a$, $a \ge 0$

(d) a dozen bagels with at least one of each kind?

Choose one of each – 8 of the bagels are determined

Bagel 1 2 3 4 5 6 7 8

$* \mid * \mid * \mid * \mid * \mid * \mid * \mid *$

Need to select 4 – repetitions allowed.

$n=8$, $\quad r=4$

$$\binom{8-1+4}{4} = \binom{11}{4}$$

(e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

Choose 3 egg bagels, ⇒ 9 more to choose

Three cases: S_0 — 0 out of 9 are salty bagels

S_1 — 1 out of 9 is a salty bagel

S_2 — 2 out of 9 "

∴ Total $= S_0 + S_1 + S_2$ (by sum rule)

	Egg	Salty	Free choice from 7 types (other than salty)	
S_0	3	0	9	$\binom{7-1+9}{9}$
S_1	3	1	8	$\binom{7-1+8}{8}$
S_2	3	2	7	$\binom{7-1+7}{7}$

$$\therefore \text{Total} = \binom{15}{6} + \binom{14}{6} + \binom{13}{6} = 9724$$

Alternative Solution:

choices with atleast 3 egg bagels $= \binom{8-1+9}{9} = \binom{16}{9} = \binom{16}{7}$

choices with at least 3 egg and 3 salty bagels $= \binom{8-1+6}{6} = \binom{13}{6}$

∴ # Choices with at least three egg bagels and no more than two salty bagels $= \binom{16}{7} - \binom{13}{6} = 9724$

Permutations of indistinguishable objects

Example: # of strings obtained by permuting the letters of the word "DISCR" is 5!

$5 \times 4 \times 3 \times 2 \times 1 = 5! = P(5,5)$

choices

$\underline{\ 5\ }\quad \underline{\ 4\ }\quad \underline{\ 3\ }\quad \underline{\ 2\ }\quad \underline{\ 1\ }$

Alternative calculation:

D — can be placed in one of five positions in $\binom{5}{1}$ ways

I — can be placed in one of the remaining 4 spots in $\binom{4}{1}$ ways

S — " " 3 spots in $\binom{3}{1}$

C — " " 2 spots in $\binom{2}{1}$

R — " " 1 spot in $\binom{1}{1}$

By product rule, total # of ways

$$= \binom{5}{1}\cdot\binom{4}{1}\cdot\binom{3}{1}\binom{2}{1}\cdot\binom{1}{1} = \frac{5!}{\cancel{4!}\,1!}\ \frac{\cancel{4!}}{\cancel{3!}\,1!}\ \frac{\cancel{3!}}{\cancel{2!}\,1!}\ \frac{\cancel{2!}}{1!\,1!}\ \frac{1!}{1!\,0!} = 5!$$

Example 2: calculate the # of strings obtained by permuting the letters of the word "DISCRETE".

Letter-count	# ways to choose spots	# spots left
D − 1	$\binom{8}{1}$	7
I − 1	$\binom{7}{1}$	6
S − 1	$\binom{6}{1}$	5
C − 1	$\binom{5}{1}$	4
R − 1	$\binom{4}{1}$	3
E − 2 →	$\binom{3}{2}$	1
T − 1	$\binom{1}{1}$	0

$$\therefore \text{\# strings} = \binom{8}{1}\cdot\binom{7}{1}\cdot\binom{6}{1}\cdot\binom{5}{1}\binom{4}{1}\cdot\binom{3}{2}\cdot\binom{1}{1}$$

$$= \frac{8!}{\cancel{7!}\,1!}\cdot\frac{\cancel{7!}}{\cancel{6!}\,1!}\;\frac{\cancel{6!}}{\cancel{5!}\,1!}\;\frac{\cancel{5!}}{\cancel{4!}\,1!}\cdot\frac{\cancel{4!}}{\cancel{3!}\,1!}\;\frac{\cancel{3!}}{1!\,2!}\cdot\frac{1!}{1!\,0!}$$

$$= \frac{8!}{2!}$$

Theorem 3, Section 5.5

The number of permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k such that $n_1+n_2+...+n_k = n$,

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_k!}$$

Distributing objects into boxes

Many counting problems can be solved by enumerating the ways objects can be placed into boxes, where the order of placing objects within a box does not matter.

- objects
  - Distinguishable (DO)
  - indistinguishable (IO)
- boxes
  - Distinguishable (DB)
  - indistinguishable (IB)

There are four possibilities. we will look at placing distinguishable objects into distinguishable boxes (DODB) and indistinguishable objects into distinguishable boxes (IODB).

Distinguishable objects into distinguishable boxes (DODB)

Example: count the number of 5-card poker hands for 4 players in a game. Assume that a standard deck of cards is used.

player 1 — $\binom{52}{5}$ 47 cards left

player 2 — $\binom{47}{5}$ 42 left

player 3 — $\binom{42}{5}$ 37 left

player 4 — $\binom{37}{5}$ 32 left.

$$\therefore \text{Total \# of hands} = \binom{52}{5}\cdot\binom{47}{5}\cdot\binom{42}{5}\cdot\binom{37}{5}$$

$$= \frac{52!}{\cancel{47!}\,5!}\cdot\frac{\cancel{47!}}{\cancel{42!}\,5!}\cdot\frac{\cancel{42!}}{\cancel{37!}\,5!}\cdot\frac{\cancel{37!}}{32!\,5!}$$

$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}$$

If the unused cards, 32 of them, are considered to be given to an imaginary player 5, then $5+5+5+5+32=52$ and theorem 4 is applicable.

Theorem 4.

The number of ways to distribute n distinguishable objects into k distinct boxes so that n_i objects are placed in box i, i=1, ..., k, and $n_1+...+n_k = n$, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Indistinguishable objects into distinguishable boxes (IODB)

Similar to combinations with repetitions. For example, consider the previously solved science fair problem.

Three schools -- A, B and C -- are competing for a grand prize in a science fair competition. There are two judges. Each judge, anonymously, recommends one of the two schools. Calculate the number of ways the judges recommend the schools.

Here, we can view

n -- schools as distinguishable boxes
r -- anonymous recommendations as indistinguishable objects

n boxes => n-1 bars separate them

r objects => when placed you have some sequence of r objects and n-1 bars

Therefore, the problem is similar to constructing (n-1+r)-bit strings with exactly r zeros (objects).

Theorem 5.
The number of ways to place r indistinguishable objects into n distinct boxes is

$$\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$$

How many terms are there in the expansion of $(x+y)^n$? $n+1$

How many terms are there in the expansion of $(x+y+z)^n$?

$$\underbrace{\overbrace{(x+y+z)}(x+y+z)\cdots\overbrace{(x+y+z)}^{IO}}_{n\text{ times}}$$

3 DBs – x, y and z

$$\binom{3-1+n}{n} = \binom{n+2}{n} = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2!}$$

$$\binom{a}{b} = \binom{a}{a-b}, \quad 0 \le b \le a$$

$(x+y+z+w)^{10}$ has $\binom{4-1+10}{10} = \binom{13}{3}$ terms

3/21/12

Each term has a multinomial coefficient. For example,

$\frac{10!}{5!\,2!\,2!}$ is the coeff. of the term $x^5 \cdot y^2 w^1 z^2$.

$\underbrace{}$

denoted $\binom{10}{5,2,1,2}$

Problems from section 6.5

11. How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels.

P	N
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0

$n = 2$ — boxes

$r = 8$

$$\binom{2-1+8}{8} = \binom{9}{8} = 9$$

15. Consider the equation $x_1+x_2+x_3+x_4+x_5 = 21$, where x_i's are nonnegative integers. How may solutions are feasible if

there are no additional constrainsts on the values of x_i's.

21 can be considered as 21 1's.

example sol, $x_1 = 5$ $x_2 = 5$ $x_3 = 5$ $x_4 = 5$ $x_5 = 1$

$\underset{x_1}{\boxed{5}}$ $\underset{x_2}{\boxed{5}}$ $\underset{x_3}{\boxed{5}}$ $\underset{x_4}{\boxed{5}}$ $\underset{x_5}{\boxed{1}}$

DB = 5 ← x_i's   ID — 21 ← twenty one 1's

$$\# \text{ of Solutions} = \binom{5-1+21}{21} = \binom{25}{21} = \binom{25}{4}$$

(a) $x_1 \geq 1$

$$\underset{x_1}{\boxed{1}} \quad \underset{x_2}{\boxed{}} \quad \underset{x_3}{\boxed{}} \quad \underset{x_4}{\boxed{}} \quad \underset{x_5}{\boxed{}} \qquad \binom{5-1+20}{20}$$

20 free choices

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$.

$$\underset{x_1}{\boxed{2}} \quad \boxed{2} \quad \boxed{2} \quad \boxed{2} \quad \underset{x_5}{\boxed{2}}$$

$$\binom{5-1+11}{11} = \binom{15}{11}$$

free choices 11

(c) $0 \leq x_1 \leq 10$?

# of Solutions w/o restriction is $\binom{5-1+21}{21}$.

↑ counts cases where $x_1 \geq 11$

# of cases with $x_1 \geq 11$: Set aside 11 1's for x_1.

Distribute the remaining 10 1's in $\binom{5-1+10}{10} = \binom{14}{10}$ ways.

$\therefore$ Required count $\binom{5-1+21}{21} - \binom{5-1+10}{10}$

Note Title

10f types of croissants $- 6 = n$

need to choose 24 croissants with

plain ≥ 1 $\boxed{1}$

cherry ≥ 2 $\boxed{2}$

choc. ≥ 3 $\boxed{3}$

almond ≥ 1 $\boxed{1}$

apple ≥ 2 $\boxed{2}$

broccoli ≤ 3 $\boxed{\ }$

9 pre-allocation

15 choices left $\binom{n-1+15}{15} = \binom{6-1+15}{15} = \binom{20}{15} = \binom{20}{5}$

at least 4 broccoli $\boxed{4}$

$\Rightarrow$ 11 choices $\binom{6-1+11}{11} = \binom{16}{11} = \binom{16}{5}$ $\leftarrow$ invalid cases

$\therefore$ Required count $= \binom{20}{15} - \binom{16}{11}$

at least 1 plain,
2 cherry
3 choc
1 almond
2 apple
+ 0–15 broccoli

at least 1 plain,
2 cherry
3 choc
1 almond
2 apple
+ 4–15 broccoli

Problems on Binomial Coefficients, Section 6.4

22(a) $\binom{n}{r}\cdot\binom{r}{k} = \binom{n}{k}\cdot\binom{n-k}{r-k}$

Scenario: from a group of n people, form a committee of r people, k of whom will be executive members.
There are two methods to form such a committee.

① Choose committee $\binom{n}{r}$

Choose executives $\binom{r}{k}$.

Total ways $\binom{n}{r}\cdot\binom{r}{k}$

② Choose exec. committee $\binom{n}{k}$

Choose the rest $\binom{n-k}{r-k}$

Total ways $\binom{n}{k}\cdot\binom{n-k}{r-k}$

23 $\binom{n+1}{\alpha} = (n+1)\binom{n}{\alpha-1}/\alpha$.

Hint: $\alpha\binom{n+1}{\alpha} = (n+1)\binom{n}{\alpha-1}$

(use 22 a to give comb. arguments)

28a. $\binom{2n}{2} = 2\cdot\binom{n}{2} + n^2$

Consider the formation of a committee of 2 from a group of n men and n women.
There are C(2n,2) such committees.

Alternative approach: Count the committees that have only men, only women, or one man and one woman.

Can have – both men ———— $\binom{n}{2}$

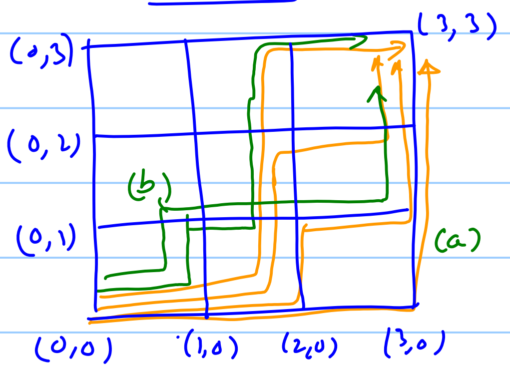
both women ———— $\binom{n}{2}$

or a man and a woman $\rightarrow \binom{n}{1}\cdot\binom{n}{1} = n\cdot n = n^2$

Total ways $= \binom{n}{2}+\binom{n}{2}+n^2 = RHS$

33.

4x4 grid



All shortest paths have 6 steps

rowsteps = 3

col. steps = 3

Total Steps = 6

$$\binom{6}{3} = \frac{6\cdot 5\cdot 4}{3!} = 20$$

Each shortest-path can be modeled as a 6-bit binary string with 3 1s and 3 0's. 1-represents a row step, 0-a col. step

E.g., path a is given by 111000, (b) by 101100.

[VB], 1.36

Show that $\binom{2n}{2} - n = 2n(n-1)$

Scenario: n couples. each person shakes hands with everyone else. # of handshakes among non-spouse pairs?

RHS: 2n people each person has (2n-2) non-spouse partners

$$\frac{2n(2n-2)}{2} = \frac{4n(n-1)}{2} = 2n(n-1)$$

Division rule. Each handshake is counted by two people

[VB], 1.30 (Similar to [KR] Section 6.4, 28a)

$$\binom{m+n}{2} - \binom{m}{2} - \binom{n}{2} = mn$$

Scenario: m men, n women.

Form a committee of 1 man and 1 woman

14. n is a positive int. S.T.

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n}$$

Let $1 \leq k \leq \frac{n}{2}$ an integer.

is $\binom{n}{k-1} < \binom{n}{k}$?

$$\binom{n}{k-1} = \frac{n!}{(n-k+1)!\,(k-1)!} \qquad \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{(n-k)!\,k!}}{\frac{n!}{(n-k+1)!\,(k-1)!}} = \frac{\cancel{n!}}{(n-k)!\,k!} \cdot \frac{(n-k+1)!\,(k-1)!}{\cancel{n!}}$$

$$= \frac{(n-k+1)!}{(n-k)!} \cdot \frac{(k-1)!}{k!}$$

$(n-k+1)! = (n-k+1)\cdot(n-k)!$

$k! = k \cdot (k-1)!$

$$= \frac{n-k+1}{k}$$

if $k \leq \frac{n}{2}$ $\qquad n-k+1 \geq n - \frac{n}{2} + 1 = \frac{n}{2}+1$

$$\frac{n-k+1}{k} \geq \frac{n/2 + 1}{n} > 1$$

$$\therefore \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

$$\parallel \qquad \parallel \qquad \qquad \parallel$$

$$\binom{n}{n} < \binom{n}{n-1} < \cdots < \binom{n}{\lceil \frac{n}{2} \rceil}$$

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{n-\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$$

24. p is a prime. Show that $p \mid \binom{p}{k}$, $1 \le k \le p-1$

$$\binom{p}{k} = \frac{p!}{(p-k)!\ k!}$$

Since $p \nmid (p-k)!$ and $p \nmid k!$

but $p \mid p!$,

$$p \mid \frac{p!}{(p-k)!\ k!}$$

$$\therefore p \mid \binom{p}{k}$$