A random variable (RV) is a real-valued function defined on the sample space $\Omega$ of a RE.

Let $X$ be a RV defined on the sample space $\Omega$. If $\omega \in \Omega$, then $X(\omega)$ is a real number.

Example: Consider the RE of 3 fair coin flips.

Let $X$ be a RV denoting the number of heads observed.

$X(HHH) = 3 \quad X(HTH) = 2 \quad X(HTT) = X(THT) = 1 \quad X(TTH) = X(TTT) = 0$

$X$ partitions $\Omega$ and takes 0, 1, 2, or 3

$P(X = a) =$ prob. $X$ takes value $a$

$P(X = 3) = \frac{1}{8} \quad P(X = 2) = \frac{3}{8} \quad P(X = 1) = \frac{3}{8} \quad P(X = 0) = \frac{1}{8}$

Prob. distribution of $X$

$P(X \geq 2) = P(X = 2 \lor X = 3) = P(X = 2) + P(X = 3)$

$= \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.

$P(X = 4) = 0 \quad P(X = 2.5) = 0 \quad P(X = -1) = 0 \quad P(-100) = 0$

$P(X \geq 0) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$
Example:

Two players play a game by flipping a fair coin one or two times. If the first flip results in a head, then the game stops. Otherwise, the coin is flipped just one more time, and the game stops regardless of the outcome. If the game ends in a head, then player 1 loses $1 to player 2. Otherwise, player 1 wins $3 from player 2.

Give the sample space. If $X$ is a RV denoting the winnings by player 1, give the values $X$ takes.

$$\Omega = \{ H, TH, TT \}$$

$$X(H) = -1$$

$$X(TH) = -1$$

$$X(TT) = 3$$

$$P(H) = \frac{1}{2}$$

$$P(TH) = P(T \cap H) = P(T) \cdot P(H | T) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(TT) = P(T) \cdot P(T | T) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
Bernoulli trials

A random event (RE) that has exactly two outcomes—success with prob. \( p \) and failure with prob. \( 1-p \)—is a Bernoulli trial.

1. Coin flip—Head (success) with prob. \( p \)

2. If (B) then \{\} else \{}

\[ P(\text{success}) + P(\text{failure}) = p + (1-p) = 1 \]

Fair coin: \( P(\text{heads}) = p = 1/2 \).

Discrete uniform probability distribution

Consider a RE with finite sample space, \( \Omega \), such that \( |\Omega| = n \).

If each sample point is equally likely, then the probability distribution is uniform.

Each sample point occurs with probability \( 1/n \).
Binomial probability distribution

Consider a sequence of \( n \) independent Bernoulli trials, each with \( p \) as the probability of success.

Let \( X \) = the number of heads observed in \( n \) coin flips

Then,

\[
P(\text{exactly } k \text{ out of } n \text{ flips resulted in heads}) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}
\]

A coin flip

\[
\begin{array}{c}
\text{p (head)} \\
\text{1-p (tail)}
\end{array}
\]

\[
\frac{p}{q} + \frac{1-p}{q} = 1
\]

\[
(p+q)^n = \binom{n}{0} p^n + \binom{n}{1} p^{n-1} q + \binom{n}{2} p^{n-2} q^2 + \cdots + \binom{n}{n} q^n
\]

\[
\begin{align*}
\text{P}(X=0) & = \binom{n}{0} p^n \\
\text{P}(X=1) & = \binom{n}{1} p^{n-1} q \quad \cdots \\
\text{P}(X=k) & = \binom{n}{k} p^k q^{n-k} \\
\text{P}(X=n) & = \binom{n}{n} q^n
\end{align*}
\]
Discrete Random Variables

Let $X$ defined on a set take values $x_1, x_2, \ldots, x_n$.

$p(x = x_i) = P(X = x_i)$ is the prob. that $X$ takes value $x_i$.

$p(x_1), p(x_2), \ldots, p(x_n)$ define a prob. mass function (pmf) for RV $X$.

$P(X \leq x_i) =$ prob. that $X$ takes values $x_1, x_2, \ldots, x_i$.

$F(x_i) =$ cumulative distribution function (CDF).

$F(x_i) = P(X \leq x_i) = p(x = x_1) + p(x = x_2) + \cdots + p(x = x_i)$

$= p(x_1) + p(x_2) + \cdots + p(x_i) = \sum_{k=1}^{i} p(x_k).$

More specifically, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

$F(x_3) = p(x_1) + p(x_2) + p(x_3) = F(x_2) + p(x_3)$

$F(x_i) = F(x_{i-1}) + p(x_i), \quad i = 1, \ldots, n$

$F(a) = 0$ if $a < x_1$,

$F(b) = P(X \leq b) = p(x_1) + p(x_2)$

$F(c) = p(x \leq c) = P(X \leq x_n) = p(x_1) + \cdots + p(x_n) = 1$

$P(X \leq a) = P(X \leq x_1) + P(X \leq x_2)$

$F(e) = P(X \leq e) = P(X \leq x_{n-1}) = p(x_1) + p(x_2) + \cdots + p(x_{n-1}) = 1 - p(x_n)$
Example: Consider the RE of 3 fair coin flips.

\[ \Omega = \{ HHH, HHT, HTH, HTT, TTH, THT, THT, TTT \} \]

Let \( X \) be a RV denoting the # of heads observed.

\[ X(\text{HHH}) = 3 \quad X(\text{HHT}) = X(\text{HTH}) = X(\text{THH}) \]

\[ X(\text{HTT}) = 1 = X(\text{THT}) = X(\text{TTT}) \]

\[ X(\text{TTT}) = 0 \]

\[ P(0) = P(0) = \frac{1}{8} \quad P(1) = \frac{3}{8} \quad P(2) = \frac{3}{8} \quad P(3) = \frac{1}{8} \]

\[ F(0) = P(0) = \frac{1}{8} \quad F(1) = F(0) + P(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2} \]

\[ F(2) = F(1) + P(2) = \frac{1}{2} + \frac{3}{8} = \frac{7}{8} \quad \text{if } \alpha < 0 \]

\[ F(3) = F(2) + P(3) = \frac{7}{8} + \frac{1}{8} = 1 \quad P(0) = 0 \]

\[ F(4) = F(3) + P(4) = 1 + 0 = 1 \quad F(5) = 1 \]
Properties of pmf, \( p() \)

Let \( T \) = image of \( x = \{ x_1, x_2, \ldots \} \)

(i) \( p(x_i) \geq 0 \) \( \forall \) all \( x_i \)

(ii) \( T \) is finite or countably infinite

(iii) \( \sum_{x_i \in T} p(x_i) = 1 \)

Properties of CDF, \( F() \)

(i) \( F \) is a nondecreasing function.

\[ \Rightarrow \text{If } x < y, \text{ then } F(x) \leq F(y) \]

(ii) \( \lim_{x \to +\infty} F(x) = 1 \)

(iii) \( \lim_{x \to -\infty} F(x) = 0 \)

Let \( a < b \).

\[ P(a < x \leq b) = F(b) - F(a) \]

\[ F(b) = \frac{P(a < x \leq b) + P(x \leq a)}{F(a)} \]

\[ \therefore F(b) - F(a) = P(a < x \leq b) \]
Expectation and Variance

The pmf and CDF describe a RV completely. The expectation and variance describe the RV concisely.

Expectation is denoted $E(X)$ or $\mu$. Variance is denoted $V(X)$ or $\sigma^2$.

Let RV $X$ take values $x_1, x_2, ..., x_n$ with nonzero probability.

$$\mu = E(X) = \sum_{k=1}^{n} x_k p(x_k)$$

**Special case:**

$$p(x_1) = p(x_2) = \cdots = p(x_n) = \frac{1}{n}$$

$$E(X) = \frac{1}{n} \cdot x_1 + \frac{1}{n} \cdot x_2 + \cdots + \frac{1}{n} \cdot x_n$$

$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\sigma^2 = V(X) = \sum_{k=1}^{n} (x_k - \mu)^2 \cdot p(x_k)$$

Standard deviation, $\sigma = \sqrt{V(X)}$

Coefficient of variation $CV = \frac{\sigma}{\mu}$

**Example:** Let $X$ be the RV indicating the number of heads in a 3 fair-coin flop RE.

$$p(0) = \frac{1}{8}, \quad p(1) = \frac{3}{8}, \quad p(2) = \frac{3}{8}, \quad p(3) = \frac{1}{8}$$

$$\mu = E(X) = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$

$$= 0 + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3 + 6 + 3}{8} = \frac{12}{8} = 1.5$$

$$V(X) = (0 - 1.5)^2 \cdot p(0) + (1 - 1.5)^2 \cdot p(1) + (2 - 1.5)^2 \cdot p(2) + (3 - 1.5)^2 \cdot p(3)$$

$$= 0.75$$

$$\sigma = \sqrt{0.75} \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{0.75}}{1.5}$$