100 Statements to Prove for CS 2233

The idea of this document is to provide a list of statements that any student of CS 2233 (Discrete Mathematical Structures) at UTSA ought to be able to prove. The point of being able prove 100 statements is to avoid simple memorization. The proofs cover many, but not all, areas of discrete mathematics, focusing more on the topics in the UTSA course.

Note: there are than 100 things to prove in this document.

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Propositional Logic

Assume $p$ and $q$ are propositional variables.

1. Prove $p \to q$ is equivalent to each of the following statements:
   - $\neg p \lor q$
   - $\neg q \rightarrow \neg p$
   - $\neg (p \land \neg q)$

2. Prove $p \oplus q$ is equivalent to each of the following statements:
   - $(p \lor q) \land (\neg p \lor \neg q)$
   - $(p \land \neg q) \lor (\neg p \land q)$
   - $\neg p \oplus \neg q$

3. Prove $p \leftrightarrow q$ is equivalent to each of the following statements:
   - $(p \rightarrow q) \land (q \rightarrow p)$
   - $(p \land q) \lor (\neg p \land \neg q)$
   - $(p \lor \neg q) \land (\neg p \lor q)$
   - $\neg (p \oplus q)$

4. Prove $\neg (p \land q)$ is equivalent to $\neg p \lor \neg q$.

5. Prove $\neg (p \lor q)$ is equivalent to $\neg p \land \neg q$.

6. Prove $(p \land (p \rightarrow q)) \rightarrow q$ is a tautology.
7. Prove \((p \land (p \rightarrow q)) \land \neg q\) is a contradiction.

8. Prove \((q \land (p \rightarrow q)) \rightarrow p\) is a contingency.

9. Prove \(((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)\) is a tautology.

10. Write a statement equivalent to the following truth table and prove its equivalence. Also write a succinct English equivalence.

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Predicate Logic

Assume the domain is the integers.

1. Prove \(\exists x(x^2 = x + 2)\).

2. Prove \(\neg \forall x(x \geq 0)\).

3. Prove \(\exists x \forall y(xy = y)\).

4. Prove \(\exists x \forall y(xy = x)\).

5. Prove \(\forall x \exists y(x < y)\).

6. Prove \(\neg \exists x \forall y(x \leq y)\).

7. For each of the following statements, either prove it or disprove it.

\[ \exists x \forall y(x < y^2) \]
\[ \forall y \exists x(x < y^2) \]
8. Prove \((\exists x \forall y P(x, y)) \rightarrow (\forall y \exists x P(x, y))\).

9. Prove \(\neg \exists x Q(x)\) is equivalent to \(\forall x (\neg Q(x))\).

10. Prove \(\neg \forall x Q(x)\) is equivalent to \(\exists x (\neg Q(x))\).

Proofs Involving Inequalities

Assume the domain is the real numbers. Assume all variables are universally quantified.

1. Prove \(x^2 \geq 0\).

2. Prove \((x^2 + y^2) \geq 2xy\).

3. Prove \(x < y\) implies \(x < z \lor z < y\).

4. Prove \(x + y < 0\) implies \(x < 0 \lor y < 0\).

5. Prove \(x = 1 \lor y = 1\) implies \(xy < x + y\).

6. Prove \(x > 2 \land y > 2\) implies \(xy > x + y\).

Assume \(a\), \(b\), and \(c\) are real numbers such that \(a < b < c\).

7. Prove \(0 < b - a\) and \(b - a < c - a\).

8. Prove \(1/(b - a) > 1/(c - a)\).

9. Prove \(1/(a - b) < 1/(a - c)\).

10. Prove \(ab - ac - b^2 + bc > 0\).

Sets
Assume that $A$ and $B$ are finite sets.

1. Prove $A = (A \cap B) \cup (A - B)$.

2. Prove $A = (A \cup B) - (B - A)$.

3. Prove $\emptyset = (A - B) \cap (B - A)$.

4. Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

5. Prove $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

6. Prove $|A \cup B| = |A| + |B| - |A \cap B|$.

7. Prove $|A| \leq |A \cup B|$.

8. Prove $|A| \geq |A \cap B|$.


10. Provide a counterexample to $|A| \geq |B - A|$.

**Even and Odd Numbers**

Assume the domain is the integers.

Let $E = \{x \mid x \in \mathbb{Z}$ and $x$ is even$\}$.

Let $O = \{x \mid x \in \mathbb{Z}$ and $x$ is odd$\}$.

Definition: An integer $x$ is even iff there exists an integer $y$ such that $x = 2y$.

Definition: An integer $x$ is odd iff there exists an integer $y$ such that $x = 2y + 1$.

1. Prove $42 \in E$.

2. Prove $-13 \in O$.

3. Prove $2x - 2 \in E$.

4. Prove $2x - 3 \in 0$.

5. Prove $E$ and $O$ are infinite sets.

6. Prove $E \cap O = \emptyset$.

7. Prove $E \cup O = \mathbb{Z}$.

8. Prove the following facts about adding even and odd numbers:
An even number plus an even number is even.
An odd number plus an odd number is even.
An even number plus an odd number is odd.

9. Prove the following facts about multiplying even and odd numbers.
   An even number times an even number is even.
   An odd number times an odd number is odd.
   An even number times an odd number is even.

10. Prove $x^2 + x \in E$.

**Big-Oh**

Assume $n$ is a positive integer.

1. Prove $n + 1000$ is $O(n)$.

2. Prove $n + 1000$ is $O(n^2)$.

3. Prove $(n + 1)^2$ is $O(n^2)$.

4. Prove $(n - 1)^2$ is not $O(n)$.

5. Prove $(n + 1)^3$ is $O(n^3)$.

6. Prove $(n - 1)^3$ is not $O(n^2)$.

7. Prove $\sum_{i=1}^{n} i$ is $O(n^2)$.

8. Prove $\sum_{i=1}^{n} i$ is not $O(n)$.

9. Prove $\sum_{i=1}^{n} i^2$ is $O(n^3)$.

10. Prove $\sum_{i=1}^{n} i^2$ is not $O(n^2)$.

**Mathematical Induction with Sums**

Assume $n$ is a positive integer and $r$ is a real number.
1. Prove \( \sum_{i=1}^{n} (a + ib) = an + b \sum_{i=1}^{n} i \)

2. Prove \( \sum_{i=1}^{n} i = n(n + 1)/2 \).

3. Prove \( \sum_{i=1}^{n} (2i - 1) = n^2 \).

4. Prove \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \).

5. Prove the following:
\[
\sum_{i=0}^{n} (ar^i) = \begin{cases} 
\frac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\
(n + 1)a & \text{if } r = 1 
\end{cases}
\]

6. Prove \( \sum_{i=1}^{n} 1/(i(i + 1)) = 1 - 1/(n + 1) \).

7. Prove \( \sum_{i=1}^{n} i > n^2/2 \) without using the exact formula for \( \sum_{i=1}^{n} i \).

8. Prove \( \sum_{i=1}^{n} i < (n + 1)^2/2 \) without using the exact formula for \( \sum_{i=1}^{n} i \).

9. Prove \( \sum_{i=1}^{n} i^2 > n^3/3 \) without using the exact formula for \( \sum_{i=1}^{n} i^2 \).

10. Prove \( \sum_{i=1}^{n} i^2 < (n + 1)^3/3 \) without using the exact formula for \( \sum_{i=1}^{n} i^2 \).

**Mathematical Induction with Exponentials, Factorials and Fibonacci Numbers**

Assume \( n \) is a nonnegative integer.

1. Prove \( 2^n \) is even when \( n > 0 \).
2. Prove $n < 2^n$.

3. Find the smallest value for $k$ such that $n^2 < 2^n$ when $n > k$. Prove it. Use this result to prove that $2^n$ is not $O(n)$.

4. Prove $n!$ is even when $n > 1$.

5. Find the smallest value for $k$ such that $n < n!$ when $n > k$. Prove it.

6. Find the smallest value for $k$ such that $n^2 < n!$ when $n > k$. Prove it. Use this result to prove that $n!$ is not $O(n)$.

Let $F_n$ be the $n$th Fibonacci number. Note $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ when $n > 1$.

7. Determine when $F_n$ is even or odd and prove this by mathematical induction.

8. Find the smallest value for $k$ such that $n < F_n$ when $n > k$. Prove it.

9. Find the smallest value for $k$ such that $n^2 < F_n$ when $n > k$. Prove it. Use this result to prove that $F_n$ is not $O(n)$.

10. Determine when the following inequalities are true and prove your hypotheses by mathematical induction.

\[
\begin{align*}
2^{n/2} &< F_n \\
F_n &< 2^n \\
2^n &< n!
\end{align*}
\]

**Recursive Definitions**

1. Prove $f_0 = 2$ and $f_n = f_{n-1}$ imply $f_n = 2$.

2. Prove $f_0 = 0$ and $f_n = 2 + f_{n-1}$ imply $f_n = 2n$.

3. Prove $f_0 = 2$ and $f_n = 1 + f_{n-1}$ imply $f_n = 2 + n$.

4. Prove $f_0 = 1$ and $f_n = 2f_{n-1}$ imply $f_n = 2^n$.

5. Prove $f_0 = 1$ and $f_n = nf_{n-1}$ imply $f_n = n!$.

6. Prove $f_0 = 0$ and $f_n = 2n - 1 + f_{n-1}$ imply $f_n = n^2$. 

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7. Prove $f_0 = 1$ and $f_n = 1 + f_{n-1}/2$ imply $f_n < 2$.

8. Prove $f_0 = 1$ and $f_n = -f_{n-1}$ imply $f_n = (-1)^n$.

9. Prove $f_1 = 0$ and $f_n = \log(n/(n-1)) + f_{n-1}$ imply $f_n = \log n$. The base of the logarithm does not matter.

10. Prove that the following definition implies $f_n = \lfloor \log_2 n \rfloor$:

$$
    f_n = \begin{cases} 
    f_n = 0 & \text{if } n = 1 \\
    f_n = f_{n-1} & \text{if } n > 1 \text{ and } n \text{ is odd} \\
    f_n = 1 + f_{n/2} & \text{if } n > 1 \text{ and } n \text{ is even} 
    \end{cases}
$$

Relations

Let $R$ and $S$ be relations on a set $A$.

1. For each of the following statements, either prove it or show a counterexample:
   - If $R$ is reflexive, then $R \circ R$ is also reflexive.
   - If $R$ is symmetric, then $R \circ R$ is also symmetric.
   - If $R$ is antisymmetric, then $R \circ R$ is also antisymmetric.
   - If $R$ is transitive, then $R \circ R$ is also transitive.

2. For each of the following statements, either prove it or show a counterexample:
   - If $R$ and $S$ are reflexive, then $R \cup S$ is also reflexive.
   - If $R$ and $S$ are symmetric, then $R \cup S$ is also symmetric.
   - If $R$ and $S$ are antisymmetric, then $R \cup S$ is also antisymmetric.
   - If $R$ and $S$ are transitive, then $R \cup S$ is also transitive.

3. For each of the following statements, either prove it or show a counterexample:
   - If $R$ and $S$ are reflexive, then $R \cap S$ is also reflexive.
   - If $R$ and $S$ are symmetric, then $R \cap S$ is also symmetric.
   - If $R$ and $S$ are antisymmetric, then $R \cap S$ is also antisymmetric.
   - If $R$ and $S$ are transitive, then $R \cap S$ is also transitive.

4. For each of the following relations, prove whether or not the relation is reflexive, symmetric, antisymmetric and/or transitive.

$$
\begin{align*}
\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x = y\} & \quad 0 \\
\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x \neq y\} & \quad 0 \\
\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x < y\} & \quad 0
\end{align*}
$$
\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x \leq y\}

\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x^2 = y\}

\{(x, y) \mid x \in \mathbb{Z} \land y \in \mathbb{Z} \land x \text{ divides } y\}

5. For each of the following statements, either prove it or show a counterexample:
   If \( R \) is reflexive, then the transitive closure of \( R \) is also reflexive.
   If \( R \) is symmetric, then the transitive closure of \( R \) is also symmetric.
   If \( R \) is antisymmetric, then the transitive closure of \( R \) is also antisymmetric.
   If \( R \) is transitive, then the transitive closure of \( R \) is also transitive.

**Graphs**

Assume \( v \) is a positive integer.

1. If \( G \) is a connected, undirected graph with \( v \) vertices, prove that \( G \) has at least \( v - 1 \) edges. Hint: Prove that any undirected graph with \( e \) edges (where \( e < v - 1 \) ) has at least \( n - e \) components.

2. If \( G \) is a strongly connected, directed graph with \( v \) vertices, prove that \( G \) has at least \( v \) edges.

3. If \( G \) is an undirected graph with \( v \) vertices, prove that \( G \) has at most \( v(v - 1)/2 \) edges.

4. If \( G \) is a directed graph with \( v \) vertices, prove that \( G \) has at most \( v^2 \) edges.

Assume \( e \) is a positive integer.

5. If \( G \) is a undirected graph with \( e \) edges, prove that \( G \) contains at least \( \sqrt{2e} \) vertices.

6. If \( G \) is a directed graph with \( e \) edges, prove that \( G \) contains at least \( \sqrt{e} \) vertices.

**Trees**

Assume \( n \) is a positive integer.

1. Prove that a binary tree with \( n \) nodes has \( n - 1 \) edges.

2. Prove that the height of a binary tree with \( n \) nodes is at most \( n - 1 \).

3. Prove that the height of a binary tree with \( n \) nodes is at least \( \lceil \log_2 n \rceil \).

Assume \( h \) is a nonnegative integer.

4. Prove that a binary tree with height \( h \) contains at least \( h + 1 \) nodes.
5. Prove that a binary tree with height $h$ contains at most $2^{h+1} - 1$ nodes.