\[
\frac{n}{2} < 2^n
\]

\[
\begin{align*}
1 & < 2 \\
2 & < 4 \\
3 & < 8 \\
4 & < 16
\end{align*}
\]

\[n \geq 1 \text{ implies } n < 2^n \]??

ture for \( n = 1 \)  

basis step  

\[ n = 1 < 2 = 2^1 \]

induction step:  

\[
\begin{align*}
&P(1) \rightarrow P(2) \\
&P(2) \rightarrow P(3) \\
&P(3) \rightarrow P(4)
\end{align*}
\]

induction basis proves:

Assume \( k \geq 1 \land k < 2^k \)

want to show:

\[ k+1 < 2^{k+1} \]

Direct Proof of:

\[ k \geq 1 \land k < 2^k \text{ implies } k+1 < 2^{k+1} \]
Short proof

for inductive step

\[ n \geq 1 \implies n < 2^n \]

\[ k \geq 1 \land k < 2^k \implies k + 1 \leq k + k < 2^k + 2^k = 2^{k+1} \]

Done
\[
\frac{n}{1} = \frac{2n-1}{1} = \sum_{i=1}^{n} (2i-1) = \sum_{i=1}^{n} \frac{n^2}{1} = \sum_{i=1}^{n} (2i-1) + 2n-1
\]

\[\sum_{i=1}^{5} (2i-1) = \text{sum of first } n \text{ positive odd integers}\]

\[n^2 = (n-1)^2 + 2n-1\]
Assume \( k \geq 1 \) and \( \sum_{i=1}^{k} (2i-1) = k^2 \).

Want to show \( \sum_{i=1}^{k+1} (2i-1) = (k+1)^2 \).

We know \( (k+1)^2 = k^2 + 2k + 1 \) and \( (k+1)^2 = k^2 + 2k + 1 \).

So \( \sum_{i=1}^{k} (2i-1) = k^2 \) and \( (k+1)^2 = \sum_{i=1}^{k} (2i-1) + (2k+1) \).

We know \( 2k+1 = 2(k+1) - 1 \).

So \( (k+1)^2 = \sum_{i=1}^{k+1} (2i-1) + (2k+1) \) and \( 2k+1 = 2(k+1) - 1 \).

Therefore, \( (k+1)^2 = \sum_{i=1}^{k+1} (2i-1) \).
use MI to show

\[ n \geq 6 \implies P(n) \]

\[ n \geq 4 \implies n^2 < n! \]
MI of $n \geq 4$ implies $n^2 < n!$

**basis:** $n = 4$

$n = 4$ implies $n^2 = 16 < n! = 24$

**induction:**

Assume: $k \geq 4$ and $k^2 < k!$

implies $k! > k^2$

Want to show: $(k+1)^2 < (k+1)!$

\[
(k+1)^2 = k^2 + 2k + 1 < k! + 2k + 1
\]

\[
(k+1)! = (k+1)k! > (k+1)k^2 = k^2 + k^2
\]

maybe we can show

2$k + 1 < k^3$

2$k + 1 < 3k < k^2 < k^3$
Proof of \( k \geq 4 \land k^2 < k! \)

implies

\[(k+1)^2 < (k+1)! \]

\( k \geq 4 \land k^2 < k! \) implies

\[(k+1)^2 = k^2 + 2k + 1 \]

< \( k^2 + 3k \)

< \( k^2 + k^2 \)

< \( k^2 + k^2 = k^3 + k^2 \)

= \((k+1)k^2 \)

< \((k+1)!k! \)

= \((k+1)! \)
Then suppose \( n \geq 5 \) implies \( a \geq 2 \).

We want to show

\[ \forall n \in \mathbb{N}, \quad 0 < 2 - n \leq \frac{2^2}{2} - n + 2 = 1 + n - 2 \leq 25 \]

for \( n \geq 5 \) implies \( n^2 \geq 25 \).
MI for \( n \geq 5 \) implies \( n^2 < 2^n \)

**Basis:** Show true for \( n = 5 \)

\( n = 5 \) implies \( n^2 = 25 < 2^5 = 32 \)

**Induction:** Show by inductive hypothesis

Assume \( n \geq 5 \) \( k^2 < 2^k \) implies

Want to show \((k+1)^2 < \frac{2^{k+1}}{2^{k+1}}\)

\((k+1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1\)

\(2^{k+1} = 2 \cdot 2^k = 2^{k+1} > k^2 + k^2\)

Maybe we can show

\( k^2 + 2k + 1 < k^2 + k^2 = 2k^2 \)

\( k^2 + 2k + 1 < k^2 + 3k < k^2 + k^2 \)
Direct Proof of

\[ k \geq 5 \land k^2 < 2^k \]
implies
\[ (k+1)^2 < 2^{k+1} \]

\[ k \geq 5 \land k^2 < 2^k \]
implies
\[ (k+1)^2 = k^2 + 2k + 1 < k^2 + 3k < k^2 + k^2 < 2^k + 2^k = 2^{k+1} \]
<table>
<thead>
<tr>
<th>$n$</th>
<th>$\sum$</th>
<th>$\frac{n^2}{2}$</th>
<th>$\frac{n^2/2}{2}$</th>
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<td>1</td>
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</tr>
<tr>
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<td>3</td>
<td>4</td>
<td>$2$</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>$\frac{9}{2}$</td>
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<tr>
<td>4</td>
<td>10</td>
<td>16</td>
<td>$4.5$</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>25</td>
<td>$8$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$12.5$</td>
</tr>
</tbody>
</table>

$k = 1$ and $\sum_{i=1}^{k} i \geq \frac{k^2}{2}$ imply

\[
\frac{(k+1)^2}{2} = \frac{k^2 + 2k + 1}{2} = \frac{k^2}{2} + k + \frac{1}{2}
\]

\[
\sum_{i=1}^{k} i < k + \frac{1}{2} + \sum_{i=1}^{k} i < k + 1 + \sum_{i=1}^{k} i
\]

\[
\text{used assume!}
\]

\[
\text{recursion: } \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)
\]
What is \( 1 + 2^1 + 2^2 + 2^3 + \ldots + 2^n \)?

\[
\sum_{i=0}^{n} 2^i
\]

\[
\begin{array}{cccc}
\frac{n}{0} & \frac{2^n}{1} & \frac{1+2^1+\ldots+2^n}{3} & \frac{2^{n+1}-1}{3} \\
1 & 2 & 3 & 3 \\
2 & 4 & 7 & 7 \\
3 & 8 & 15 & 15 \\
4 & 16 & 31 & 31 \\
5 & 32 & 63 & 63 \\
\end{array}
\]

\( n \geq 0 \) implies \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \)

Want to show by MI
\[ n \geq 0 \text{ implies } \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \]

**Basis**: Show for \( n = 0 \)
\[ n=0 \text{ implies } \sum_{i=0}^{0} 2^i = 1 = 2^{0+1} - 1 = 1 \]

**Induction**: Assume \( k \geq 0 \land \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \)
\[ \text{Want to show } \sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1 \]

Note that
\[ \sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i \]
\[ 2^{k+2} - 1 = 2^{k+1} + 2^{k+1} - 1 \]
Proof:

\[ k \geq 0 \land \sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \]

implies

\[ \sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i = 2^{k+1} + 2^{k+1} - 1 = 2^{k+2} - 1 \]
$n \geq b$ implies $P(n)$

**Basis:** Show $P(b)$ and $P(b+1)$

**Induction:** Prove

\[ k \geq b+1 \land P(b) \land P(b+1) \land \ldots \land P(k) \]

implies

\[ P(k+1) \]
\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 \\
2 & 2 & 4 & 5 \\
3 & 3 & 8 & 6 \\
4 & 5 & 16 & 7 \\
5 & 8 & 32 & 7 \\
6 & 13 & 64 & 7 \\
7 & 21 & 128 & 7
\end{array}
\]

Use MI (Strong Induction)

\( n \geq 6 \) implies \( p(n) \)

\( n \geq 0 \) implies \( F_n < 2^n \)

Def. \( F_n \)

\[
F_0 = 0 \quad F_1 = 1 \\
F_n = F_{n-1} + F_{n-2} \\
F_{k+1} = F_k + F_{k-1}
\]

\[
2^n = 2 \cdot 2^{n-1} \\
2^k = 2 \cdot 2^{k-1} \\
2^{k+1} = 2 \cdot 2^k \\
F_{k+1} = F_k + F_{k-1}
\]
MI (Strong Induction) for
\[ n \geq 0 \text{ implies } F_n < 2^n \]

**Basis:** \( n = 0 \) implies \( F_0 = 0 < 2^0 = 1 \)
\[ n = 1 \text{ implies } F_1 = 1 < 2^1 = 2 \]

**Induction:**
Assume: \( k \geq 1 \) \( \land F_0 < 2^0 \land F_1 < 2^1 \land \ldots \land F_{k-1} < 2^{k-1} \land F_k < 2^k \)
implies \( F_{k+1} < 2^{k+1} \)

Want to show: \( F_{k+1} < 2^{k+1} \)

\[ F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} \]

\[ 2^{k+1} = 2^k + 2^k \]

maybe we can show \( 2^{k-1} < 2^k \)

obviously true
Proof of:

\[ k \geq 1 \land F_0 < 2^0 \land \ldots \land F_k < 2^k \]
implies
\[ F_{k+1} < 2^{k+1} \]

\[ k \geq 1 \land F_0 < 2^0 \land \ldots \land F_k < 2^k \]
implies
\[ F_{k+1} = F_k + F_{k-1} < 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1} \]
For $n = 0$ implies $F_n < 2^n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$F_n$</th>
<th>$(\sqrt{2})^n$</th>
<th>$2^{n/2}$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>0</td>
<td>1</td>
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</tr>
<tr>
<td>5</td>
<td>5</td>
<td>32</td>
<td>32</td>
</tr>
</tbody>
</table>

For $n = 4$ implies $F_{2n} > 2^n$
\[ n \geq 4 \implies F_{2n} > 2^n \]

**Basis:** show for \( n = 4 \)

Proof: \( n = 4 \) implies \( F_{2n} = 21 > 2^4 = 16 \)

**Induction:**

Assume \( k \geq 4 \) and \( F_{2k} > 2^k \)

Want to show \( \frac{F_{2(k+1)}}{F_{2k+2}} > 2^{k+1} \)

Note all \( F_n \geq 0 \)

\[ 2^{k+1} = 2^k + 2^k = 2 \times 2^k \]

\[ F_{2k+2} = F_{2k+1} + F_{2k} = F_{2k} + F_{2k} + F_{2k-1} \]

Proof: \( k \geq 4 \) and \( F_{2k} > 2^k \)

implies

\[ F_{2k+2} = F_{2k+1} + F_{2k} = F_{2k} + F_{2k} + F_{2k-1} \]

\[ > 2^k + 2^k + F_{2k-1} \geq 2^k + 2^k = 2^{k+1} \]