Induction

Every program has at least one bug and can be shortened by at least one instruction—from which, by induction, it is evident that every program can be reduced to one instruction that does not work. (Ken Arnold)

Mathematical Induction

Principle of Mathematical Induction

Mathematical induction proves a predicate is true for all integers greater than some number.

The Principle of Mathematical Induction: To prove \( n \geq b \) implies \( P(n) \), do the following steps:

1. Basis Step: Prove \( P(b) \) is true.
2. Induction Step: Prove \( k \geq b \land P(k) \) implies \( P(k + 1) \).

In combination, the basis and induction steps prove \( \forall n(n \geq b \rightarrow P(n)) \), or in other words, \( n \geq b \) implies \( P(n) \).

Throughout, the domain is the integers.

Template for Proofs

1. Write statement to prove: \( n \geq b \) implies \( P(n) \) substituting for \( b \) and \( P(n) \).
2. Show basis step. Prove that \( P(b) \) is true.
3. Show induction step by direct proof.
   (a) Write inductive hypothesis. Assume \( k \geq b \) and \( P(k) \).
   (b) Write intended conclusion \( P(k + 1) \).
   Substitute \( n \) with \( (k + 1) \).
   (c) Prove \( P(k + 1) \).
   A key part of the proof is using \( P(k) \).
   You may also need to use \( k \geq b \).
Example 1

\[ n \geq 1 \implies \sum_{i=1}^{n} (2i - 1) = n^2 \]

Basis \((n = 1)\): \(\sum_{i=1}^{1} (2i - 1) = 1 = 1^2\)

Induction \((n > 1)\):

Assume \(k \geq 1\) and \(\sum_{i=1}^{k} (2i - 1) = k^2\)

Want to show \(\sum_{i=1}^{k+1} (2i - 1) = (k+1)^2\)

Proof: We know \((k+1)^2 = k^2 + 2k + 1\) and \(\sum_{i=1}^{k+1} (2i - 1) = \text{last term} + \text{rest of terms} + 2(k+1) - 1 + \sum_{i=1}^{k} (2i - 1)\)

\[ = 2(k+1) - 1 + k^2 + 2k + 1 \]
\[ = 2k + 1 + (k+1)^2 \]
\[ = (k+1)^2 \]

Example 2

\[ n \geq 1 \implies n < 2^n \]

Basis \((n = 1)\): \(1 < 2^1 = 2\)

Induction \((n > 1)\):

Assume \(k \geq 1\) and \(k < 2^k\)

Want to show \(k + 1 < 2^{k+1}\)

Proof: We know \(2^{k+1} = 2^k + 2^k\).

\(k < 2^k\) implies \(2^{k+1} = 2^k + 2^k > k + k\)

\(k \geq 1\) implies \(k + k \geq k + 1\)

\(k + 1 \leq k + k\) and \(k + k < 2^{k+1}\) imply \(k + 1 < 2^{k+1}\)

Example 3

\[ n \geq 1 \implies \sum_{i=1}^{n} i > n^2/2 \]

Basis \((n = 1)\): \(\sum_{i=1}^{1} i = 1 > 1^2/2 = 1/2\)

Induction \((n > 1)\):

Assume \(k \geq 1\) and \(\sum_{i=1}^{k} i > k^2/2\)

Want to show \(\sum_{i=1}^{k+1} i > (k + 1)^2/2\)

Proof Parts: \((k + 1)^2/2 = (k^2 + 2k + 1)/2\)

\[ \text{and} \sum_{i=1}^{k+1} i = k + 1 + \sum_{i=1}^{k} i \]

How to use inductive hypothesis?
Example 4

\( n \geq 5 \) implies \( n^2 < 2^n \)  Why 5?

**Basis** \((n = 5)\): \( 5^2 = 25 < 2^5 = 32 \)

**Induction** \((n > 5)\):

Assume \( k \geq 5 \) and \( k^2 < 2^k \)

Want to show \((k + 1)^2 < 2^{k+1} \)

**Proof Parts:**

\((k + 1)^2 = k^2 + 2k + 1 \)  
and \( 2^{k+1} = 2^k + 2^k \)

How to use inductive hypothesis?

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**Strong Induction**

*Strong Induction* is a more general form of mathematical induction. It also proves a predicate is true for all integers greater than some number.

Strong Induction has two steps.

1. **Basis**: Prove \( P(b) \) is true.
2. **Induction**: Prove \( k \geq b \land P(b) \land \ldots \land P(k) \) imply \( P(k + 1) \).

As with mathematical induction, the basis and induction steps prove \( n \geq b \) implies \( P(n) \).

The basis often includes more than one base case.

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Example 1

Let \( F_i \) be the \( i \)th Fibonacci number.

\( n \geq 0 \) implies \( F_n < 2^n \)

**Basis** \((n \in \{0, 1\})\): \( F_0 = 0 < 2^0 = 1 \) and \( F_1 = 1 < 2^1 = 2 \)

**Induction** \((n > 1)\):

Assume \( k \geq 1 \) and \( F_0 < 2^0, \ldots, F_k < 2^k \)

Want to show \( F_{k+1} < 2^{k+1} \)

Prove this using \( F_{k+1} = F_k + F_{k-1} \) and \( 2^{k+1} = 2^k + 2^k \) and \( F_k < 2^k \) and \( F_{k-1} < 2^{k-1} \).

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Example 2

\( n \geq 12 \) implies there exists \( i \in \mathbb{N} \) and \( j \in \mathbb{N} \) such that \( 4i + 5j = n \)

**Basis** \((n \in \{12, 13, 14, 15\})\): How?

**Induction** \((n > 15)\):

Assume \( k \geq 15 \) and there are combinations for 12, 13, 14, 15, \ldots, \( k \)

Want to show there is a combination for \( k + 1 \)

**Proof Sketch:** Add one more 4 to \( k - 3 \) combo.

Because \( k \geq 15 \) then \( k - 3 \geq 12 \) has a combination by the inductive hypothesis.
Example 3

Harmonic Numbers: Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$
Also, let $\lg n = \log_2 n$

$n \geq 1$ implies $H_n > (\lg n)/3$

Basis ($n = 1$): $H_1 = 1 > (\lg 1)/3 = 0$

Induction ($n > 1$):

Assume $k \geq 1$ and
$H_1 > (\lg 1)/3, \ldots, H_k > (\lg k)/3$

Want to show $H_{k+1} > (\lg (k+1))/3$

Example 3 Continued

Proof Parts:

The first half (or more) of the terms of $H_{k+1}$ is greater than
$\frac{\lg \frac{k+1}{2}}{3} = \frac{\lg (k + 1) - 1}{3}$

The last third (or more) of the terms of $H_{k+1}$ is greater than or equal to $1/3$.
$(k + 1)/3$ terms are each $\geq 1/(k + 1)$.

$$\left\lceil \frac{k + 1}{2} \right\rceil + \left\lceil \frac{k + 1}{3} \right\rceil \leq k + 1$$