Counting

Basic principles for counting the number of elements in a set.

1. Count each element once.
2. Don't count an element more than once.
3. Determine how elements differ, and apply the appropriate rule.

Notation:
S, S_1, S_2, ... are sets
$|S|$ = the number of elements in S.

The Sum Rule

Conditions:

- $S = S_1 \cup S_2$
- $|S_1| = n_1$
- $S_1 \cap S_2 = \emptyset$
- $|S_2| = n_2$

Conclusion: $|S| = n_1 + n_2$.

Example:
S = set of alphanumeric characters
S_1 = set of digits, and S_2 = set of letters
$S = S_1 \cup S_2$, and $S_1 \cap S_2 = \emptyset$
$|S_1| = 10$, and $|S_2| = 26$ (ignoring case)
Therefore, $|S| = 10 + 26 = 36$

The Product Rule

Conditions:

- $S = S_1 \times S_2$
- $|S_1| = n_1$
- $|S_2| = n_2$

Conclusion: $|S| = n_1 * n_2$.

Example:

$S =$ two digit numbers, no leading zeroes

$S_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$S_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$|S_1| = 9$ and $|S_2| = 10$

Therefore, $|S| = 9 * 10 = 90$

The Inclusion-Exclusion Principle

Conditions:

- $S = S_1 \cup S_2$
- $|S_1| = n_1$
- $|S_1 \cap S_2| = m$
- $|S_2| = n_2$

Conclusion: $|S| = n_1 + n_2 - m$.

Example:

$S =$ 2-digit numbers with a 9, no leading 0's

$S_1 =$ two digit numbers beginning with 9

$S_2 =$ two digit numbers ending with 9

$S = S_1 \cup S_2$, $|S_1 \cap S_2| = 1$, $|S_1| = 10$, and $|S_2| = 9$

Therefore, $|S| = 10 + 9 - 1 = 18$

Generalizations

Generalized Sum Rule:

- $S = S_1 \cup S_2 \cup \ldots \cup S_k$
- if $i \neq j$, then $S_i \cap S_j = \emptyset$
- $|S_1| = n_1$, $|S_2| = n_2$, $\ldots$, $|S_k| = n_k$

Conclusion: $|S| = \sum_{i=1}^{k} n_i$

Generalized Product Rule:

- $S = S_1 \times S_2 \times \ldots \times S_k$
- $|S_1| = n_1$, $|S_2| = n_2$, $\ldots$, $|S_k| = n_k$

Conclusion: $|S| = \prod_{i=1}^{k} n_i$

Problem structure:
Many problems require multiple rules.
How many strings of letters of length 4 or less?

The Pigeonhole Principle

Conditions:

- $S = S_1 \cup S_2 \cup \ldots \cup S_k$
- $|S| = N$

Conclusion:
At least one subset has at least $\lceil N/k \rceil$ elements.

Example 1:
Out of 50 people, how many were born in the same month?
$S = 50$ with 12 subsets
At least one month has at least $\lceil 50/12 \rceil = 5$ people born in that month.

Example 2:
A drawer contains black, brown, and white socks.
How many socks ensure two of the same color?
There are 3 subsets: black, brown, and white.
Want one subset with at least 2 elements.
Need to find N so that $\lceil N/3 \rceil = 2$.