

# Permutations

Conditions:

- $|S_1| = n$
- $S = \{(x_1, x_2, \dots, x_r) \mid$
- $x_1 \in S_1, x_2 \in S_1, \dots, x_r \in S_1,$
- $\text{if } i \neq j, \text{ then } x_i \neq x_j\}$

Conclusion:

$$|S| = P(n, r) = n(n-1)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

Each element of  $S$  is an  $r$ -permutation out of  $n$  elements.

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## Permutation Examples

Number of 3-letter initials, no letter repeated.

$$n = 26, r = 3$$

$$P(26, 3) = 26(25)(24) = \frac{26!}{(26-3)!} = \frac{26!}{23!}$$

Number of ways to select and rank favorite 5 days of the year.

$$n = 365, r = 5$$

$$\begin{aligned} P(365, 5) &= 365(364)(363)(362)(361) \\ &= \frac{365!}{(365-5)!} = \frac{365!}{360!} \end{aligned}$$

## Combinations

Conditions:

- $|S_1| = n$
- $S = \{\{x_1, x_2, \dots, x_r\} \mid$
- $x_1 \in S_1, x_2 \in S_1, \dots, x_r \in S_1,$
- $\text{if } i \neq j, \text{ then } x_i \neq x_j\}$

Conclusion:  $|S| = C(n, r) = \frac{n!}{r!(n-r)!}$

Each element of  $S$  is an  $r$ -combination out of  $n$  elements.

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## Combination Examples

Number of 3-letter sets.

$$n = 26, r = 3$$

$$C(26, 3) = \frac{26!}{3!(26-3)!} = \frac{26!}{3!23!} = \frac{26(25)(24)}{1(2)(3)}$$

Number of bit strings of length 10 with two 1s and eight 0s.

$$n = 10, r = 2 \text{ or } r = 8$$

$$C(10, 2) = C(10, 8) = \frac{10!}{2!8!} = \frac{10(9)}{1(2)}$$

## Binomial Coefficients

Often,  $C(n, r)$  is written as a binomial coefficient  $\binom{n}{r}$ . They occur as coefficients in the expansion of  $(a + b)^n$ .

Binomial Theorem:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$

Proof: The coefficient of  $x^{n-k} y^k$  is the number of ways  $y$  can be chosen  $k$  times from  $n$  terms. This is the number of  $k$ -combinations out of  $n$ .

Corollary:

$$\sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^n \binom{n}{k} 1^{n-k} x^k = (1 + x)^n$$

It follows that:  $\sum_{j=0}^n \binom{n}{k} = (1 + 1)^n = 2^n$

$$\sum_{k=0}^n \binom{n}{k} (-1)^k = (-1 + 1)^n = 0^n = 0$$

$$\sum_{k=0}^n \binom{n}{k} 2^k = (2 + 1)^n = 3^n$$

## Pascal's Identity

Theorem: 
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Proof: Let  $T$  be a set with  $n+1$  elements.

Let  $a \in T$  and  $S = T - \{a\}$ .

A subset of  $T$  of size  $k$  either contains  $a$  and  $k-1$  elements from  $S$  or omits  $a$  and contains  $k$  elements from  $S$ . (sum rule)

The number of subsets from  $S$  of size  $k-1$  and  $k$  is  $\binom{n}{k-1} + \binom{n}{k}$

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## Pascal's Triangle

$$\begin{array}{cccccc}
 & & \binom{0}{0} & & & 1 \\
 & & \binom{1}{0} & \binom{1}{1} & & 1 & 1 \\
 & & \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & 1 & 2 & 1 \\
 & & \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & 1 & 3 & 3 & 1 \\
 & & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & 1 & 4 & 6 & 4 & 1
 \end{array}$$