Sequences and Summations

A sequence is a function from a subset of $\mathbf{Z}$, typically $\mathbf{Z}^+$ or $\mathbf{N}$, to some other set.
Can be denoted by $f(n) = y$ or $a_n = y$.
Exs: $f(n) = a_n = n^2$, $f(n) = a_n = 2^n$

A summation sums the terms in a sequence.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

$$\sum_{i=0}^{5} 2^i = 1 + 2 + 4 + 8 + 16 + 32 = 63$$

$$\sum_{i=2}^{6} i^2 = 4 + 9 + 16 + 25 + 36 = 90$$

Types of Sequences and Summations

$a_n = c + dn$ is an arithmetic progression.
$a_n = c\,r^n$ is a geometric progression.
A geometric series sums a geom. progression.

$$\sum_{i=0}^{n} c\,r^i = c \sum_{i=0}^{n} r^i = c\,\frac{r^{n+1} - 1}{r - 1}$$

Note that $(r-1)\sum_{i=0}^{n} r^i = r^{n+1} - 1$ implies

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

Mathematical Induction

Mathematical induction is a way to prove a predicate for all integers greater than some number.

Mathematical Induction proves two propositions:
1. Basis: prove $P(1)$ is true.
2. Induction: prove $n \geq 1 \wedge P(n)$ imply $P(n+1)$.

Basis and induction imply $\forall n\, (n \geq 1 \rightarrow P(n))$.
Proof: The Well-Ordering Property states that $S \subseteq N$ implies $S = \emptyset$ or S has a minimum.
Note that $S = \{n \mid n \geq 1 \wedge \neg P(n)\} = \emptyset$.

Example 1

Predicate $P(n)$: $\sum_{i=1}^{n} (2i - 1) = n^2$

Basis (prove $P(1)$): $\sum_{i=1}^{n} (2i - 1) = 1 = n^2$

Induction: Prove $k \geq 1 \wedge P(k)$ imply $P(k+1)$

Assume $k \geq 1$ and $P(k)$: $\sum_{i=1}^{k} (2i - 1) = k^2$

Show $P(k+1)$: $\sum_{i=1}^{k+1} (2i - 1) = (k+1)^2 = k^2 + 2k + 1$

Proof:

$$\sum_{i=1}^{k+1} (2i - 1) = 2(k+1) - 1 + \sum_{i=1}^{k} (2i - 1) = 2k + 1 + k^2$$

Example 2

Predicate $P(n)$: $\sum_{i=1}^{n} i^2 > n^3/3$

Basis $P(1)$: $\sum_{i=1}^{1} i^2 = 1 > 1^3/3 = 1/3$

Induction: Prove $k \geq 1 \wedge P(k)$ imply $P(k+1)$

Assume $k \geq 1$ and $P(k)$: $\sum_{i=1}^{k} i^2 > k^3/3$

Show $P(k+1)$: $\sum_{i=1}^{k+1} i^2 > (k+1)^3/3 = (k^3+3k^2+3k+1)/3$

Proof: $\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2 > k^2 + 2k + 1 + k^3/3 =$

$$(k^3 + 3k^2 + 6k + 3)/3 > (k^3 + 3k^2 + 3k + 1)/3$$

Example 3

Predicate $P(n)$: $n^2 < 2^n$

Basis $P(5)$: $5^2 = 25 < 2^5 = 32$

Induction: Prove $k \geq 5 \wedge P(k)$ imply $P(k+1)$

Assume $k \geq 5$ and $P(k)$: $k^2 < 2^k$

Show $P(k+1)$: $(k+1)^2 < 2^{k+1}$

Proof:

$2^{k+1} = 2 \cdot 2^k > 2k^2 \geq k^2 + 5k \geq k^2 + 2k + 1$

Second Principle of Mathematical Induction

1. Basis: Prove $P(1)$ is true.
2. Induction:

   Prove $n \geq 2 \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(n-1)$ imply $P(n)$.

Example 4

Predicate $P(n)$: $\sum_{i=1}^{n} 1/i > (\log n)/3$

Basis $P(1)$: $\sum_{i=1}^{1} 1/i = 1 \geq (\log 1)/3 = 0$

Induction: Prove $P(1) \wedge \ldots \wedge P(k-1)$ imply $P(k)$

Assume $k \geq 2 \wedge P(1) \wedge \ldots \wedge P(k-1)$

Show $P(k)$: $\sum_{i=1}^{k} 1/i > (\log k)/3$

Proof Sketch (case when k is even, using $P(k/2)$):

$$\sum_{i=1}^{k} 1/i = \sum_{i=1}^{k/2} 1/i + \sum_{i=1+k/2}^{k} 1/i > \frac{\log(k/2)}{3} + \frac{k}{2}\left(\frac{1}{k}\right)$$