Sequences and Summations

A sequence is a function from a subset of $\mathbb{Z}$, typically $\mathbb{Z}^+$ or $\mathbb{N}$, to some other set. Can be denoted by $f(n) = y$ or $a_n = y$.

Exs: $f(n) = a_n = n^2$, $f(n) = a_n = 2^n$

A summation sums the terms in a sequence.

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$$

$$\sum_{i=0}^{5} 2^i = 1 + 2 + 4 + 8 + 16 + 32 = 63$$

$$\sum_{i=2}^{6} i^2 = 4 + 9 + 16 + 25 + 36 = 90$$

Types of Sequences and Summations

$a_n = c + dn$ is an arithmetic progression.

$a_n = c r^n$ is a geometric progression.

A geometric series sums a geom. progression.

$$\sum_{i=0}^{n} c r^i = c \sum_{i=0}^{n} r^i = c \frac{r^{n+1} - 1}{r - 1}$$

Note that $(r - 1) \sum_{i=0}^{n} r^i = r^{n+1} - 1$ implies

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$
Mathematical Induction

Mathematical induction is a way to prove a predicate for all integers greater than some number.

Mathematical Induction proves two propositions:
1. Basis: prove \( P(1) \) is true.
2. Induction: prove \( n \geq 1 \land P(n) \) imply \( P(n+1) \).

Basis and induction imply \( \forall n (n \geq 1 \rightarrow P(n)) \).

Proof: The Well-Ordering Property states that \( S \subseteq N \) implies \( S = \emptyset \) or \( S \) has a minimum.

Note that \( S = \{ n \mid n \geq 1 \land \neg P(n) \} = \emptyset \).

Example 1

Predicate \( P(n) \): \( \sum_{i=1}^{n} (2i - 1) = n^2 \)

Basis (prove \( P(1) \)): \( \sum_{i=1}^{n} (2i - 1) = 1 = n^2 \)

Induction: Prove \( k \geq 1 \land P(k) \) imply \( P(k + 1) \)

Assume \( k \geq 1 \) and \( P(k) \): \( \sum_{i=1}^{k} (2i - 1) = k^2 \)

Show \( P(k + 1) \): \( \sum_{i=1}^{k+1} (2i - 1) = (k + 1)^2 = k^2 + 2k + 1 \)

Proof:
\[
\sum_{i=1}^{k+1} (2i - 1) = 2(k + 1) - 1 + \sum_{i=1}^{k} (2i - 1) = 2k + 1 + k^2
\]
Example 2

Predicate $P(n)$: $\sum_{i=1}^{n} i^2 > n^3/3$

Basis $P(1)$: $\sum_{i=1}^{1} i^2 = 1 > 1^3/3 = 1/3$

Induction: Prove $k \geq 1 \land P(k)$ imply $P(k + 1)$

Assume $k \geq 1$ and $P(k)$: $\sum_{i=1}^{k} i^2 > k^3/3$

Show $P(k+1)$: $\sum_{i=1}^{k+1} i^2 > (k+1)^3/3 = (k^3 + 3k^2 + 3k + 1)/3$

Proof: $\sum_{i=1}^{k+1} i^2 = (k + 1)^2 + \sum_{i=1}^{k} i^2 > k^2 + 2k + 1 + k^3/3 = (k^3 + 3k^2 + 6k + 3)/3 > (k^3 + 3k^2 + 3k + 1)/3$

Example 3

Predicate $P(n)$: $n^2 < 2^n$

Basis $P(5)$: $5^2 = 25 < 2^5 = 32$

Induction: Prove $k \geq 5 \land P(k)$ imply $P(k + 1)$

Assume $k \geq 5$ and $P(k)$: $k^2 < 2^k$

Show $P(k + 1)$: $(k + 1)^2 < 2^{k+1}$

Proof: $2^{k+1} = 2 \cdot 2^k > 2k^2 \geq k^2 + 5k \geq k^2 + 2k + 1$
Second Principle of Mathematical Induction

1. Basis: Prove \( P(1) \) is true.

2. Induction:
   Prove \( n \geq 2 \land P(1) \land P(2) \land \ldots \land P(n-1) \)
   imply \( P(n) \).

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Example 4

Predicate \( P(n) \): \( \sum_{i=1}^{n} 1/i > (\log n)/3 \)

Basis \( P(1) \): \( \sum_{i=1}^{1} 1/i = 1 \geq (\log 1)/3 = 0 \)

Induction: Prove \( P(1) \land \ldots \land P(k - 1) \) imply \( P(k) \)

Assume \( k \geq 2 \land P(1) \land \ldots \land P(k - 1) \)

Show \( P(k) \): \( \sum_{i=1}^{k} 1/i > (\log k)/3 \)

Proof Sketch (case when \( k \) is even, using \( P(k/2) \)):

\[
\sum_{i=1}^{k} 1/i = \sum_{i=1}^{k/2} 1/i + \sum_{i=1+k/2}^{k} 1/i > \frac{\log(k/2)}{3} + \frac{k}{2} \left( \frac{1}{k} \right)
\]