Matrices

A *matrix* is a rectangular array of numbers.

Matrices are used to represent the relationship from one set to another set (or the same set).

An $m \times n$ matrix has m rows and n columns.

Element a_{ij} in matrix $\mathbf{A}$ is the number in the ith row and jth column.

$\begin{bmatrix} 5 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ is a 2×3 matrix.

Matrix Addition

If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, then $\mathbf{A}+\mathbf{B} = \mathbf{C}$ is an $m \times n$ matrix, where $c_{ij} = a_{ij} + b_{ij}$

$$\begin{bmatrix} 5 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 6 \\ 3 & -2 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 6 \\ 0 & -2 & -3 \end{bmatrix}$$

Matrix Multiplication

If $\mathbf{A}$ is a $m \times k$ matrix, and $\mathbf{B}$ is a $k \times n$ matrix, then $\mathbf{A}\,\mathbf{B} = \mathbf{C}$ is an $m \times n$ matrix, where c_{ij} is the ith row of $\mathbf{A}$ times the jth column of $\mathbf{B}$.

$$c_{ij} = \sum_{q=1}^{k} a_{iq}\, b_{qj} = a_{i1}\, b_{1j} + \cdots + a_{ik}\, b_{kj}$$

If $\mathbf{A}$ is $n \times n$, $\mathbf{A}^2 = \mathbf{A}\,\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}\,\mathbf{A}\,\mathbf{A}$, ...

The Boolean product of zero-one matrices is:

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$$

Matrix Multiplication Examples

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ is undefined.}$$

Matrix Multiplication Algorithm

procedure *matrix_multiply*
(**A**: $m \times k$ matrix,
B: $k \times n$ matrix)
create an $m \times n$ matrix **C**
for $i := 1$ **to** m
for $j := 1$ **to** n
$c_{ij} := 0$
for $q := 1$ **to** k
$c_{ij} := c_{ij} + a_{iq}b_{qj}$
return C

If **A** and **B** are $n \times n$, then *matrix_multiply*(**A**, **B**) performs n^3 multiplications.

Example Application (Markov Chains)

One application of matrices is for calculating probabilities of states over time (Markov chains).

Example: How well will students remember to study for the exam?

States: A student either remembers (state 1) or forgets (state 2) during a given hour.

Transitions: If a student remembers this hour, there is a 10% chance of forgetting the next hour. If a student forgets this hour, there is a 25% chance of remembering the next hour.

Represent this with the matrix $\begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix}$

Rows 1 and 2 represent being in state 1 or 2 for a given hour.

Columns represent states in the following hour.

The probabilities for two hours from now are:

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.25 & 0.75 \end{bmatrix} = \begin{bmatrix} 0.835 & 0.165 \\ 0.4125 & 0.5875 \end{bmatrix}$$

Example Application (Path Finding)

Suppose a building has 4 rooms, with doors between rooms 1 and 2, 2 and 3, 3 and 4, and 4 and 1.

Represent this with the matrix $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Each 1 represents a move from one room to another.

The possible ways to do two moves are:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$