

Simple Finite Probability

Sample Space S : The set of possible outcomes.

Event E : A subset of the sample space.

Example: Rolling two dice. Let E = sum of 7.

$|S| = 36$, 6 for 1st die times 6 for 2nd die.

$E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

If each outcome is equally likely, and S is finite, then the probability of an event is: $p(E) = \frac{|E|}{|S|}$

Finite Probability Examples

Probability that 3-letter initial has no letter repeated. $|E| = P(26, 3)$, $|S| = 26^3$

$$p(E) = \frac{|E|}{|S|} = \frac{P(26, 3)}{26^3}$$

Probability that a bit string of length 10 has two 1s and eight 0s. $|E| = C(10, 2)$, $|S| = 2^{10}$

$$p(E) = \frac{|E|}{|S|} = \frac{C(10, 2)}{2^{10}}$$

Combinations of Events

Let $\bar{E} = S - E$, then $p(\bar{E}) = 1 - p(E)$.

Example: The probability that two dice do not add up to 7 is $1 - 1/6 = 5/6$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example: The probability that a bit string of length 10 has exactly 2 1s or begins with a 1 is

$$\frac{C(10, 2)}{2^{10}} + \frac{1}{2} - \frac{C(9, 1)}{2^{10}}$$

General Finite Probability

Let S be a finite or countable set.

$p(s)$ = prob. of outcome s in sample space S .

We require $p(s) \geq 0$ and $1 = \sum_{s \in S} p(s)$.

probability of event $E = p(E) = \sum_{s \in E} p(s)$.

The function p from outcomes to probabilities is called a probability distribution.

Conditional Probability and Independence

The conditional probability of E_1 given E_2 is:

$$p(E_1 | E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)}$$

Ex: the prob. that a length 10 bit string begins in 1 given exactly 2 1s is $C(9, 1)/C(10, 2)$.

E_1 and E_2 are independent if and only if

$$p(E_1 \cap E_2) = p(E_1)p(E_2)$$

Example: Two dice adding up to 7 is independent of the first die being even. $p(E_1 \cap E_2) = 3/36$, $P(E_1) = 6/36$, $P(E_2) = 18/36$.

Example: I repeatedly flip a coin until it comes up tails. The number of flips can be any positive integer, so let $S = \mathbf{Z}^+$.

$p(n) = (1/2)^n$. Each coin flip is independent so, for example:

$$p(3) = p(HHT) = p(H)p(H)p(T) = (1/2)^3$$

The probability of flipping exactly 1 or 2 heads before it comes up tails is:

$$p(E) = p(2) + p(3) = 1/4 + 1/8 = 3/8$$

Bernoulli Trials

Sample space of size 2 \equiv Bernoulli trial

Prob. of k successes in n independent Bernoulli trials = $C(n, k) p^k (1 - p)^{n-k}$

where p is the probability of success.

Example: Suppose a student has 90% chance of answering any given question correctly. What is the probability of getting exactly 9 out of 10 questions right?

$$n = 10, k = 9, p = 0.9$$

$$C(10, 9) (0.9)^9 (0.1)^1 \approx 0.39$$

Random Variables

A random variable X is a function $X : S \rightarrow \mathbf{R}$
That is, a random variable is a function from outcomes to real numbers.

The expected value of X is $E(X) = \sum_{s \in S} p(s) X(s)$

Example: What is the expected value of the product of 2 dice?

$$X(1, 1) = 1, X(1, 2) = 2, \dots, X(6, 6) = 36$$

$$p(s) = 1/36 \text{ for all outcomes}$$

$$E(X) = (1/36)(1 + 2 + \dots + 30 + 36) = 12.25$$

Probabilistic Algorithms

Many algorithms depend on randomness for their success.

A *Las Vegas* random algorithm always produces the correct answer, but the running time varies. Examples: quicksort and hash tables.

A *Monte Carlo* random algorithm can be wrong, but allowing more running time reduces error. Ex: approx. pi by randomly generating $(x, y) \in [-1, 1] \times [-1, 1]$ and testing $x^2 + y^2 \leq 1$.