Recurrences

A recurrence relation for a sequence is an equation that defines \( a_n \) by previous terms.

Example: \( a_n = a_{n-1} + a_{n-2} \)

The initial conditions specify the terms before the recurrence relation takes effect.

Example: \( a_1 = 2, \ a_2 = 3 \)

Many counting problems can be modeled by recurrence relations.

Example 1

How many bit strings of length \( n \) have no “00”? All strings begin with a 1 or a 0.

 Starts with a 1 \( \rightarrow \) remaining \( n - 1 \) bits can be any string with no 00.

 Starts with a 0 \( \rightarrow \) 2nd bit is a 1 \( \rightarrow \) remaining \( n - 2 \) bits can be any string with no 00.

2 strings of length 1 and 3 strings of length 2 without 00.
Example 2

If I add $1000/year to an account that earns 7% interest, how much money will I have in 20 years?

The next year includes the previous year’s amount with 7% interest plus $1000.

Recurrence Relation: \( a_n = 1.07 \times a_{n-1} + 1000 \)

Initial Conditions: \( a_0 = 0 \)

Solution: \( a_{20} \approx 40995.52 \)

Example 3

A password contains letters and numbers with at least 1 letter and 1 number. How many passwords of length 8 are there?

A password of length \( n \) can be formed by:
- Any letter or number followed by a length \( n-1 \) password.
- Any number followed by \( n-1 \) letters.
- Any letter followed by \( n-1 \) numbers.

Recurrence Relation:
\[ a_n = 36 \times a_{n-1} + 10 \times 26^{n-1} + 26 \times 10^{n-1} \]

Initial Conditions: \( a_1 = 0 \)

Solution: \( a_8 = 2612182842880 \)
Algorithm Examples

How many recursive calls to compute $power(x, n)$?

Recurrence Relation: $a_n = 1 + a_{n-1}$
Initial Conditions: $a_0 = 0$
Solution: $a_n = n$

How many recursive calls to compute $fast_power(x, n)$?

Recurrence Relation: $a_n = 1 + a_{n/2}$ if $n$ is even
$a_n = 1 + a_{n-1}$ if $n$ is odd
Initial Conditions: $a_1 = 0$
Solution: $a_n = \lceil \log n \rceil + (\text{number of bits in } n) - 1$

How many recursive calls to compute $stupid_fibonacci(n)$?

Recurrence Relation: $a_n = 2 + a_{n-1} + a_{n-2}$
Initial Conditions: $a_0 = 0, a_1 = 0$
Sequence = $(0, 0, 2, 4, 8, 14, 24, 40, 66, 108, \ldots)$
Proving Properties of Recurrences

1. Generate initial terms of sequence
2. Guess a formula or upper or lower bound
3. Prove by mathematical induction

Recurrence Relation: \( a_n = 1 + a_{n/2} \) if \( n \) is even
\( a_n = 1 + a_{n-1} \) if \( n \) is odd

Initial Conditions: \( a_1 = 0 \)

Sequence = \((0, 1, 2, 2, 3, 3, 4, 3, 4, 4, \ldots)\)

Predicate \( P(n) \): \( a_n \leq 2 \log n \)

Basis \( P(1) \): \( a_1 = 0 = 2 \log 1 \)

Induction: Prove \((P(1) \land \ldots \land P(k-1)) \rightarrow P(k)\)

Assume \( P(1) \ldots P(k-1) \)

Show \( P(k) \): \( a_k \leq 2 \log k \)

Proof: if \( k \) is even: \( a_k = 1 + a_{k/2} \leq 1 + 2 \log(k/2) \leq 2(1 + \log(k/2)) = 2 \log k \)

if \( k \) is odd: \( a_k = 1 + a_{k-1} = 2 + a_{(k-1)/2} \leq 2 + 2 \log((k-1)/2) = 2(1 + \log((k-1)/2)) = 2 \log(k-1) < 2 \log k \)