Recurrences

A *recurrence relation* for a sequence is an equation that defines a_n by previous terms.

Example: $a_n = a_{n-1} + a_{n-2}$

The *initial conditions* specify the terms before the recurrence relation takes effect.

Example: $a_1 = 2,\ a_2 = 3$

Many counting problems can be modeled by recurrence relations.

Example 1

How many bit strings of length n have no "00"?

All strings begin with a 1 or a 0.

Starts with a 1 $\rightarrow$ remaining $n-1$ bits can be any string with no 00.

Starts with a 0 $\rightarrow$ 2nd bit is a 1 $\rightarrow$ remaining $n-2$ bits can be any string with no 00.

2 strings of length 1 and 3 strings of length 2 without 00.

Example 2

If I add \$1000/year to an account that earns 7% interest, how much money will I have in 20 years?

The next year includes the previous year's amount with 7% interest plus \$1000.

Recurrence Relation: $a_n = 1.07 * a_{n-1} + 1000$
Initial Conditions: $a_0 = 0$

Solution: $a_{20} \approx 40995.52$

Example 3

A password contains letters and numbers with at least 1 letter and 1 number. How many passwords of length 8 are there?

A password of length n can be formed by:
Any letter or number followed by a length $n-1$ password.
Any number followed by $n-1$ letters.
Any letter followed by $n-1$ numbers.

Recurrence Relation:

$$a_n = 36 * a_{n-1} + 10 * 26^{n-1} + 26 * 10^{n-1}$$

Initial Conditions: $a_1 = 0$

Solution: $a_8 = 2612182842880$

Algorithm Examples

How many recursive calls to compute $power(x, n)$?

Recurrence Relation: $a_n = 1 + a_{n-1}$
Initial Conditions: $a_0 = 0$

Solution: $a_n = n$

How many recursive calls to compute $fast\_power(x, n)$?

Recurrence Relation: $a_n = 1 + a_{n/2}$ if n is even
$a_n = 1 + a_{n-1}$ if n is odd
Initial Conditions: $a_1 = 0$

Solution: $a_n = \lfloor \log n \rfloor + (\text{number of bits in } n) - 1$

How many recursive calls to compute $stupid\_fibonacci(n)$?

Recurrence Relation: $a_n = 2 + a_{n-1} + a_{n-2}$
Initial Conditions: $a_0 = 0, \ a_1 = 0$

Sequence $= (0, 0, 2, 4, 8, 14, 24, 40, 66, 108, \ldots)$

Proving Properties of Recurrences

1. Generate initial terms of sequence
2. Guess a formula or upper or lower bound
3. Prove by mathematical induction

Recurrence Relation: $a_n = 1 + a_{n/2}$ if n is even
$a_n = 1 + a_{n-1}$ if n is odd

Initial Conditions: $a_1 = 0$

Sequence $= (0, 1, 2, 2, 3, 3, 4, 3, 4, 4, \ldots)$

Predicate $P(n)$: $a_n \leq 2 \log n$

Basis $P(1)$: $a_1 = 0 = 2 \log 1$

Induction: Prove $(P(1) \wedge \ldots \wedge P(k-1)) \rightarrow P(k)$

Assume $P(1) \ldots P(k-1)$

Show $P(k)$: $a_k \leq 2 \log k$

Proof: if k is even: $a_k = 1 + a_{k/2} \leq 1 + 2\log(k/2) \leq 2(1 + \log(k/2)) = 2 \log k$

if k is odd: $a_k = 1 + a_{k-1} = 2 + a_{(k-1)/2} \leq 2 + 2\log((k-1)/2) = 2(1 + \log((k-1)/2)) = 2\log(k-1) < 2 \log k$