Relations

A *relation* from a set A to a set B is a subset of $A \times B$.
A *relation* on a set A is a subset of $A \times A$.

If $P(x, y)$ is a predicate with universe A, then
$\{(a, b) \mid P(a, b) \text{ is true}\}$ is a relation on A.

If R and S are relations on A, then
$R \cup S$, $R \cap S$, $R - S$, and $\overline{R}$ are relations on A.

If f is a function from A to A, then
$\{(a, b) \mid a \in A \wedge f(a) = b\}$ is a relation on A.

If A has n elements, then
a relation on A has between 0 and n^2 elements,
and there are 2^{n^2} possible relations on A.

More Examples of Relations

Numeric comparisons: `==`, `<`, `<=`, `>=`, `>`, `!=`
$\{(0, 0), (1, 1), (-1, -1), (2, 2), (-2, -2), \ldots\}$

String comparisons: `strcmp`, `strstr`
{(`"roof"`, `"proof"`), (`"set"`, `"settle"`), ...}

Functions on numbers: `log`, `sqrt`, `floor`, `abs` $\{(0, 0), (1, 1), (4, 2)$

Conversions: `atof`, `atoi`, `sprintf`
{(`"0123"`, 123), (`" -45"`, -45), ...}

Any array or sequence: $\{(0, a_0), (1, a_1), \ldots\}$
$\{(0, 1), (1, 2), (2, 4), (3, 8), (4, 16), \ldots\}$

Types of Relations

A relation R on a set A is *reflexive* $\equiv \forall x\ ((x,x) \in R)$
Examples: $=, \leq, \geq$

A relation R on a set A is *symmetric* $\equiv$
$\forall x, y\ ((x,y) \in R \rightarrow (y,x) \in R)$
Examples: $=, \neq$

A relation R on a set A is *antisymmetric* $\equiv$
$\forall x, y\ ((x,y) \in R \rightarrow (x = y \vee (y,x) \notin R))$
Examples: $=, <, \leq, \geq, >$

A relation R on a set A is *transitive* $\equiv$
$\forall x, y, z\ (((x,y) \in R \wedge (y,z) \in R) \rightarrow (x,z) \in R)$
Examples: $=, \leq, \geq, <, >$

Other Types of Relations

If R and S are relations on A, the *composite* of R and S is $\{(a,c) \mid (a,b) \in R \wedge (b,c) \in S\}$.

R^2 is the composite of R and R.
R^{n+1} is the composite of R^n and R.

The reflexive/symmetric/transitive *closure* of a relation $R = R \cup$ just enough elements to make it reflexive/symmetric/tran

A relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.

A relation is a *partial ordering* if it is reflexive, antisymmetric, and transitive.