# Sets

$x \in A$ means $x$ is an element of $A$.
$x \notin A$ means $x$ is not an element of $A$.

There are two notations for describing sets.

List. $A = \{1, 3, 5, 7, 9\}$. $B = \{2, 3, 5, 7\}$.
Set Builder. $C = \{x \mid x \in \mathbf{Z}^+ \wedge x \text{ is odd}\}$

Equality. $A = B \ \equiv \ \forall x\,(x \in A \leftrightarrow x \in B)$
This implies that $\{1, 2, 3\} = \{1, 2, 2, 3, 3, 3\}$.

Subset. $A \subseteq B \ \equiv \ \forall x\,(x \in A \rightarrow x \in B)$

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# Special Sets

$\emptyset = \{\ \}$. The empty set. $\forall x\,(x \notin \emptyset)$.
$\mathbf{Z} = \{x \mid x \text{ is an integer}\}$
$\mathbf{R} = \{x \mid x \text{ is a real number}\}$
$\mathbf{N} = \{x \mid x \text{ is an integer} \wedge x \geq 0\}$
$\mathbf{Z}^+ = \{x \mid x \text{ is an integer} \wedge x > 0\}$
$\mathbf{R}^+ = \{x \mid x \text{ is a real number} \wedge x > 0\}$
$\{0, 1, \ldots, 9\} = \{x \mid x \text{ is integer} \wedge 0 \leq x \leq 9\}$

The power set of $S$ is the set of all subsets of $S$.
The power set of $\{1, 3\} = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$.

# More Set Terminology

$|A| = n \;\equiv\; A$ is a finite set with $n$ elements.
$\mathbf{Z}$ and $\mathbf{R}$ are examples of infinite sets.

An $n$-tuple $(a_1, \ldots, a_n)$ orders $n$ elements.
A 2-tuple is called an ordered pair.
$\{1,3\} = \{3,1\}$, but $(1,3) \neq (3,1)$.

Cartesian product.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

$$\{1,3\} \times \{2,3\} = \{(1,2),(1,3),(3,2),(3,3)\}$$

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# Set Operations

Let $U$ be the universal set.

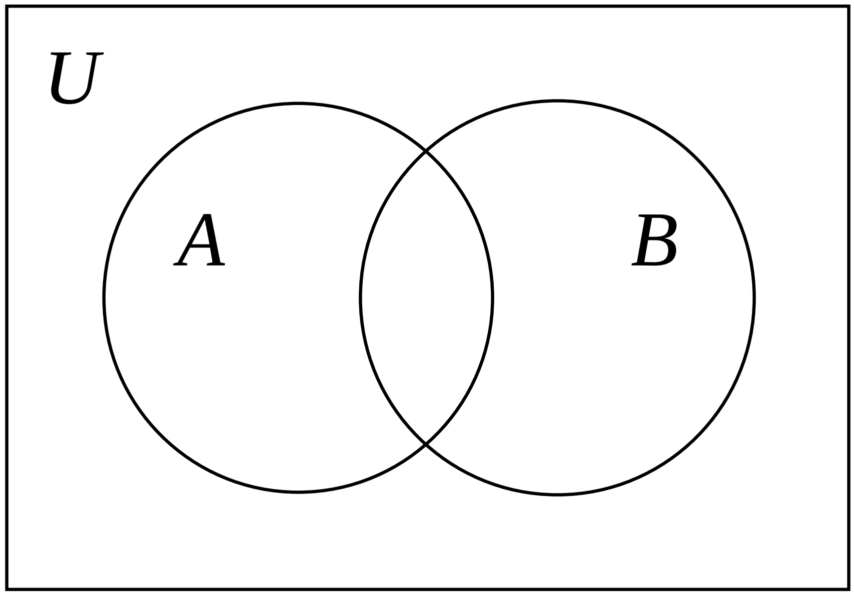
| | |
|---|---|
| union | $A \cup B = \{x \mid x \in A \vee x \in B\}$ |
| intersection | $A \cap B = \{x \mid x \in A \wedge x \in B\}$ |
| difference | $A - B = \{x \mid x \in A \wedge x \notin B\}$ |
| complement | $\overline{A} = \{x \mid x \in U \wedge x \notin A\}$ |

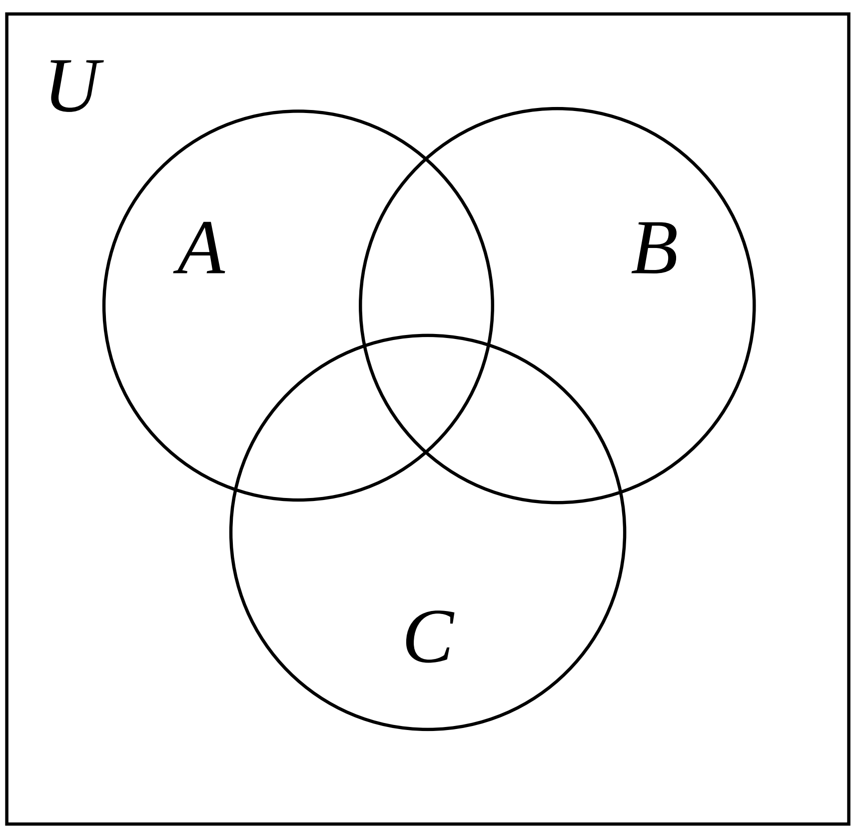
$A$ and $B$ are disjoint $\;\equiv\; A \cap B = \emptyset$

Set identities include De Morgan's laws for sets.

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Use Venn Diagrams to prove set identities.


U
A
B


U
A
B
C

# Functions

Let $A$ and $B$ be sets.

A function $f$ from $A$ to $B$ assigns exactly one element of $B$ to each element of $A$.

$f(a) = b \;\equiv\; f$ assigns $b \in B$ to $a \in A$

$f : A \rightarrow B \;\equiv\; f$ is a function from $A$ to $B$

$A$ is called the domain of $f$.

$B$ is called the codomain or range of $f$.

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# Example Functions

$+ : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$

What about minus, subtraction, multiplication, division, exponentiation, logarithm, mod?

$\lfloor x \rfloor$ is the floor of $x$, the largest integer $\leq x$.
Exs: $\lfloor 2.3 \rfloor = 2,\; \lfloor 1 \rfloor = 1,\; \lfloor -3.4 \rfloor = -4$

$\lceil x \rceil$ is the ceiling of $x$, the smallest integer $\geq x$.
Exs: $\lceil 2.3 \rceil = 3,\; \lceil 1 \rceil = 1,\; \lceil -3.4 \rceil = -3$

$\lfloor\;\rfloor : \mathbf{R} \rightarrow \mathbf{Z}$ and $\lceil\;\rceil : \mathbf{R} \rightarrow \mathbf{Z}$

# Types of Functions

$f$ is one-to-one (injective) $\equiv$
$\forall x \, \forall y \, (x \neq y \rightarrow f(x) \neq f(y))$

$f$ is strictly increasing $\equiv$
$\forall x \, \forall y \, (x < y \rightarrow f(x) < f(y)$

$f$ is onto (surjective) $\equiv$ $\forall y \, \exists x \, (f(x) = y)$

$f$ is a one-to-one correspondence (bijective) $\equiv$
$f$ is one-to-one $\wedge$ $f$ is onto

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If $f$ is a one-to-one correspondence,
then $f$ has an inverse function $f^{-1}$ such that

$$\forall x \, \forall y \, (f(x) = y \;\leftrightarrow\; x = f^{-1}(y))$$

Examples of $f : \mathbf{Z} \rightarrow \mathbf{Z}$

$f(x) = x^2$, not one-to-one, not onto

$f(x) = x^3$, one-to-one, not onto

$f(x) = \lfloor x/2 \rfloor$, not one-to-one, onto

$f(x) = \lfloor 3x/2 \rfloor$, one-to-one, not onto

$f(x) = x + 1$, one-to-one, onto, $f^{-1}(y) = y - 1$