Chapter 2: Fundamentals of the Analysis of Algorithm Efficiency

The first rule of any technology used in a business is that automation applied to an efficient operation will magnify the efficiency. The second is that automation applied to an inefficient operation will magnify the inefficiency. (Bill Gates)

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Analysis Framework

Analysis of Algorithms

- Issues:
  - Correctness. Is the algorithm correct?
  - Time Efficiency. How much time does the algorithm use?
  - Space Efficiency. How much extra space (in addition to the space needed for the input and output) does the algorithm use?
- Approaches:
  - Theoretical Analysis. Proof of correctness and big-Oh and related notations.
  - Empirical Analysis. Testing and measurement over a range of instances.

Efficiency as a Function of Input Size

Typically, more time and space are needed to run an algorithm on bigger inputs (e.g., more numbers, longer strings, larger graphs).

Analyze efficiency as a function of n=*size of input*.

- Searching/sorting. n=number of items in list.
- String processing. n = length of string(s).
- Matrix operations, n = dimension of matrix. $n \times n$ matrix has n^2 elements.
- Graph processing. n_V = number of vertices and n_E = number of edges.

Measuring Running Time

Analyze time efficiency by:

- identifying the *basic operation(s)*, the operation(s) contributing the most to running time,
- characterizing the number of times it is performed as a function of input size.

We can crudely estimate running time by $T(n) \approx c_{op} * C(n)$

- $T(n)$: running time as a function of n.
- c_{op}: running time of a single basic operation.
- $C(n)$: number of basic operations as a function of n.

Worst-Case, Best-Case, and Average-Case

algorithm $SequentialSearch(A[0..n-1], K)$
  // Searches for a value in an array
  // Input: An array A and a search key K
  // Output: The index where K is found or -1
  for $i \leftarrow 0$ **to** $n-1$ **do**
    if $A[i] = K$ **then return** i
  return -1

- Basic Operation: The comparison in the loop
- Worst-Case: n comparisons
- Best-Case: 1 comparison
- Average-Case: $(n+1)/2$ comparisons assuming each element equally likely to be searched.

Order of Growth

Typically, the basic operation count can be approximated as $c\,g(n)$, where $g(n)$ is the *order of growth*, often some "simple" function such as:

TABLE 2.1 Values (some approximate) of several functions important for analysis of algorithms

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3{\cdot}10^1$	10^2	10^3	10^3	$3.6{\cdot}10^6$
10^2	6.6	10^2	$6.6{\cdot}10^2$	10^4	10^6	$1.3{\cdot}10^{30}$	$9.3{\cdot}10^{157}$
10^3	10	10^3	$1.0{\cdot}10^4$	10^6	10^9		
10^4	13	10^4	$1.3{\cdot}10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7{\cdot}10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0{\cdot}10^7$	10^{12}	10^{18}		

Asymptotic Order of Growth

A way of comparing functions that ignores constant factors and small input sizes.

- Big-Oh $O(g(n))$: functions $\leq c\,g(n)$.

  $t(n) \in O(g(n))$ if there are positive constants c and n_0 such that $t(n) \leq c\,g(n)$ for all $n \geq n_0$

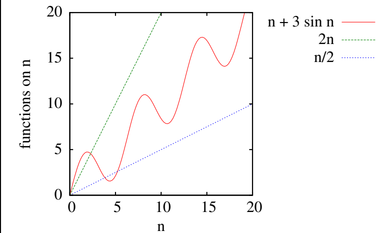
- Big-Omega $\Omega(g(n))$: functions $\geq c\,g(n)$.

  $t(n) \in \Omega(g(n))$ if there are positive constants c and n_0 such that $t(n) \geq c\,g(n)$ for all $n \geq n_0$

- Big-Theta $\Theta(g(n))$: functions $\approx c\,g(n)$.

  Both $t(n) \in O(g(n))$ and $t(n) \in \Omega(g(n))$.

Illustration of Asymptotic Order



Graph suggests (and we can prove) $n + 3\sin n$ is:
$O(n)$: $n + 3\sin n \leq 2n$ when $n \geq 3$.
$\Omega(n)$: $n + 3\sin n \geq n/2$ when $n \geq 6$.

Examples of Asymptotic Order

$t(n)$	$O(n)$	$O(n^2)$	$O(n^3)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$
$\log_2 n$	T	T	T	F	F	F
$10n + 5$	T	T	T	T	F	F
$n(n-1)/2$	F	T	T	T	T	F
$(n+1)^3$	F	F	T	T	T	T
2^n	F	F	F	T	T	T

For example, $10n + 5$ is $\Theta(n)$. Assuming $n \geq 5$:
$10n+5 \in O(n)$ because $10n+5 \leq 10n+n = 11n$.
$10n + 5 \in \Omega(n)$ because $10n + 5 \geq 10n$.

Properties of Order of Growth

- If $f_1(n)$ is order $g_1(n)$, and $f_2(n)$ is order $g_2(n)$, then $f_1(n) + f_2(n)$ is order $\max(g_1(n), g_2(n))$.
- If $b > 1$, then $\log_b n \in \Theta(\log n)$.
- Polynomials of degree $k \in \Theta(n^k)$, that is, $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0 \in \Theta(n^k)$.
- Exponential functions a^n have different orders of growth for different a's.
- order $\log n <$ order n^k where $k > 0 <$ order a^n where $a > 1 <$ order $n! <$ order n^n

Basic Asymptotic Efficiency Classes

Class	Name	Example
1	constant	access array element
$\log n$	logarithmic	binary search
n	linear	find median
$n \log n$	"n-log-n"	mergesort
n^2	quadratic	insertion sort
n^3	cubic	matrix multiplication
a^n	exponential	generating all subsets
$n!$	factorial	generating all permutations

Mathematical Analysis of Nonrecursive Algorithms

Plan for Analyzing Nonrecursive Algorithms

- Decide on parameter n indicating input size.
- Identify algorithm's basic operation(s).
- Determine worst, average, and best cases for input of size n.
- Set up a sum for the number of times the basic operation is executed.
- Simplify the sum using standard formulas and rules (see Appendix A).

Useful Summation Formulas and Rules

- $\sum_{i=1}^{n} 1 = 1 + 1 + \cdots + 1 = n \in \Theta(n)$
- $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \in \Theta(n^2)$
- $\sum_{i=1}^{n} i^k = 1 + 2^k + \cdots + n^k \approx \frac{n^{k+1}}{k+1} \in \Theta(n^{k+1})$
- $\sum_{i=1}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1} \in \Theta(a^n)$
- $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \qquad \sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$

Finding the Maximum

algorithm $MaxElement(A[0..n-1])$
  // Returns the maximum value in an array
  // Input: A nonempty array A of real numbers
  // Output: The maximum value in A
  $maxval \leftarrow A[0]$
  for $i \leftarrow 1$ **to** $n-1$ **do**
    if $A[i] > maxval$ **then**
      $maxval \leftarrow A[i]$
  return $maxval$

Basic Operation: comparison in loop

Performs $\sum_{i=1}^{n-1} 1 = n-1$ comparisons

Element Uniqueness

algorithm $UniqueElements(A[0..n-1])$
  // Returns true if all elements are different
  // Input: An array A
  // Output: Returns true if there are no duplicates
  for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
      if $A[i] = A[j]$ **then return false**
  return true

- Basic Operation: inner loop comparison
- Best/Worst-Case: from 1 to $n(n-1)/2$

$$\sum_{i=0}^{n-2} n-(i+1) = (n-1)+(n-2)+\cdots+1 = \sum_{k=1}^{n-1} k = n(n-1)/2$$

Matrix Multiplication

algorithm $MatrixMultiply(A, B : n \times n$matrices$)$
  // Multiplies matrix A times matrix B
  // Input: Two $n \times n$ matrices A and B
  // Output: Matrix $C = AB$
  for $i \leftarrow 0$ **to** $n-1$ **do**
    for $j \leftarrow 0$ **to** $n-1$ **do**
      $C[i,j] \leftarrow 0$
      for $k \leftarrow 0$ **to** $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] * B[k,j]$
  return C

Basic Operation: inner loop multiplications

Performs $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} 1 = n^3$ multiplications

Number of Bits in an Integer

algorithm $BitCount(m)$
  // Input: A positive integer m
  // Output: The number of bits to encode m
  $count \leftarrow 1$
  while $m > 1$ **do**
    $count \leftarrow count + 1$
    $m \leftarrow \lfloor m/2 \rfloor$
  return $count$

n = length of m in bits
Basic Operation: The division by 2 in loop
Performs $n-1$ operations because $n = count$.

Plan for Analyzing Recursive Algorithms

- Decide on parameter n indicating input size.
- Identify algorithm's basic operation(s).
- Determine worst, average, and best cases for input of size n.
- Set up a recurrence relation expressing the basic operation count.
- Solve the recurrence (at least determine it's order of growth).

Factorial Function

algorithm $Factorial(n)$
  // Computes $n!$ recursively
  // Input: A nonnegative integer n
  // Output: The value of $n!$
  if $n = 0$ **then return** 1
  else return $Factorial(n-1) * n$

Input Size: Use number n (actually n has about $\log_2 n$ bits)
Basic Operation: multiplication

Recurrence for Factorial Function

- Let $M(n)$ = multiplication count to compute $Factorial(n)$.
- $M(0) = 0$ because no multiplications are performed to compute $Factorial(0)$.
- If $n > 0$, then $Factorial(n)$ performs recursive call plus one multiplication.

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute} \\ Factorial(n-1)}} + \underbrace{1}_{\substack{\text{to multiply} \\ Factorial(n-1) \text{ by } n}}$$

Solving the Factorial Recurrence

Make a reasonable guess.

- Forward substitution. $M(1) = M(0) + 1 = 1$
  $M(2) = M(1) + 1 = 2$
  $M(3) = M(2) + 1 = 3$
- Backward substitution. $M(n) = M(n-1) + 1$
  $= [M(n-2) + 1] + 1 = M(n-2) + 2$
  $= [M(n-3) + 1] + 2 = M(n-3) + 3$

Prove $M(n) = n$ by mathematical induction.

- Basis: if $n = 0$, then $M(n) = M(0) = 0 = n$
- Induction: if $M(n-1) = n-1$, then
  $M(n) = M(n-1) + 1 = (n-1) + 1 = n$

Towers of Hanoi Algorithm

algorithm $Towers(n, i, j)$
  // Moves n disks from peg i to peg j
  // Input: Integers $n > 0$, $1 \le i, j \le 3$, $i \ne j$
  // Output: Specifies disk moves in correct order
  if $n = 1$ **then**
    move disk 1 from peg i to peg j
  else
    $Towers(n-1,\ i,\ 6-i-j)$
    move disk n from peg i to peg j
    $Towers(n-1,\ 6-i-j,\ j)$

Input Size: Use number n
Basic Operation: moving a disk

Recurrence for Towers of Hanoi

- Let $M(n)$ = move count to compute $Towers(n, \cdot, \cdot)$.
- $M(1) = 1$ because 1 move is needed for $Towers(1, \cdot, \cdot)$.
- If $n > 1$, then $Towers(n, \cdot, \cdot)$ performs 2 recursive calls plus one move.

$$M(n) = 2M(n-1) + 1$$

to compute $Towers(n-1, \cdot, \cdot)$ twice; to move disk n

Solving the Towers of Hanoi Recurrence

Make a reasonable guess.

- Forward substitution. $M(2) = 2M(1) + 1 = 3$
  $M(3) = 2M(2) + 1 = 7$
  $M(4) = 2M(3) + 1 = 15$
- Backward substitution. $M(n) = 2M(n-1) + 1$
  $= 2[2M(n-2) + 1] + 1 = 4M(n-2) + 3$
  $= 4[2M(n-3) + 1] + 2 = 8M(n-3) + 7$

Prove $M(n) = 2^n - 1$ by mathematical induction.

- Basis: if $n = 1$, then $M(n) = 1 = 2^n - 1$
- Induction: if $M(n-1) = 2^{n-1} - 1$, then
  $M(n) = 2M(n-1) + 1 = 2 * (2^{n-1} - 1) + 1 = (2^n - 2) + 1 = 2^n - 1$

Number of Bits in an Integer

algorithm $BitCount(n)$
  // Input: A positive integer n
  // Output: The number of bits to encode n
  if $m = 1$ **then return** 1
  else return $BitCount(\lfloor n/2 \rfloor) + 1$

Input Size: Use number n (actually n has about $\log_2 n$ bits)
Basic Operation: division by 2

Recurrence for BitCount Function

- Let $D(n)$ = division count to compute $BitCount(n)$.
- $D(1) = 0$ because no divisions are performed to compute $BitCount(1)$.
- If $n > 1$, then $BitCount(n)$ performs recursive call on $\lfloor n/2 \rfloor$ plus one division.

$$D(n) = D(\lfloor n/2 \rfloor) + 1$$

to compute $BitCount(\lfloor n/2 \rfloor)$; to compute $\lfloor n/2 \rfloor$

Solving the BitCount Recurrence

Make a reasonable guess using powers of 2.

- Forward substitution. $D(2) = D(1) + 1 = 1$
  $D(4) = D(2) + 1 = 2$
  $D(8) = D(4) + 1 = 3$
- Backward substitution. $D(n) = D(n/2) + 1$
  $= [D(n/4) + 1] + 1 = D(n/4) + 2$
  $= [D(n/8) + 1] + 2 = D(n/8) + 3$

Prove $D(2^k) = k$ by mathematical induction.

- Basis: if $k = 0$, then $D(2^k) = D(1) = 0 = k$
- Induction: if $D(2^{k-1}) = k - 1$, then $D(2^k) = D(2^{k-1}) + 1 = (k - 1) + 1 = k$

Fibonacci Function

algorithm $F(n)$
  // Computes nth Fibonacci number recursively
  // Input: A nonnegative integer n
  // Output: The nth Fibonacci number
  if $n = 0$ or $n = 1$ **then return** n
  else return $F(n-1) + F(n-2)$

Input Size: Use number n (actually n has about $\log_2 n$ bits)
Basic Operation: addition in else statement

Recurrence for Fibonacci Function

- Let $A(n)$ = addition count to compute $F(n)$.
- $A(1) = A(0) = 0$ because no additions are performed to compute $F(0)$ or $F(1)$.
- If $n > 1$, then $F(n)$ performs two recursive calls plus one addition.

$A(n) = A(n-1) + A(n-2) + 1$

to compute $F(n-1)$	to compute $F(n-2)$	to add $F(n-1)$ and $F(n-2)$

Approximating the Fibonacci Recurrence

Make a reasonable guess at a lower bound.

- Forward substitution. $A(2) = 1$, $A(3) = 2$, $A(4) = 4$, $A(5) = 7$, $A(6) = 12$.
- Backward substitution.
  $A(n) = A(n-1) + A(n-2) + 1$
  $= [A(n-2) + A(n-3) + 1] + A(n-2) + 1$
  $= 2A(n-2) + A(n-3) + 2$

Prove $A(n) \geq 2^{n/2}$ when $n \geq 4$.

- Basis: True for $A(4)$ and $A(5)$.
- Induction: $A(n) = 2A(n-2) + A(n-3) + 2$
  $\geq 2A(n-2) \geq 2 * 2^{(n-2)/2} = 2^{n/2}$

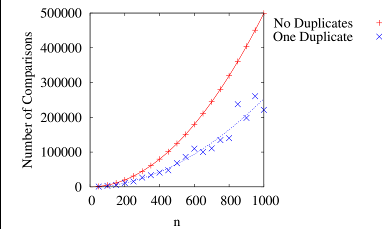
Plan for Empirical Analysis of Algorithms

- Understand the experiment's purpose.
- Decide on the metric M and the measurement unit.
- Decide on characteristics of input.
- Prepare program implementing algorithm(s) and generating a sample of inputs.
- Run the algorithm(s) on the sample and record the data.
- Analyze the data.

Empirical Analysis of UniqueElements

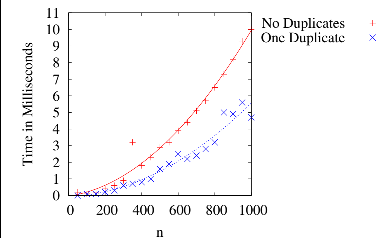
- Analyze *UniqueElements*, comparing arrays with unique elements vs. one duplicate.
- Measure number of comparisons on different input sizes, from 50 to 1000 by 50.
- For arrays with a duplicate, randomly choose a pair of positions that will have the same value. Also, run 10 times for each input size.
- Prepare `UniqueElementsExperiment.java`.
- Run program, record data, and create graph.
- Analyze the data.

Results of Empirical Analysis: Comparisons



- No duplicates line is $n(n-1)/2$.
- One duplicate line is least squares fit, resulting in $0.24 * n^2 + 17 * n - 2042 \approx n^2/4$.
- Predict: Half the time with one duplicate.

Results of Empirical Analysis: Timings



- No duplicates and one duplicate lines are:
  $-0.037 + 0.0016 * n + 8.4 \times 10^{-6} * n^2$.
  $-0.073 + 0.00047 * n + 5.2 \times 10^{-6} * n^2$.

Algorithm Visualization

Visualization of Sorting Algorithms

http://homepages.dcc.ufmg.br/~dorgival/applets/SortingPoints/SortingPoints

http://maven.smith.edu/~thiebaut/java/sort/demo.html