

Homework 11

CS 3343 – Fall 2006
Tom Bylander, Instructor

assigned November 16, 2006
due December 1, 2006

Your solutions must be submitted as a document to WebCT.

1. (100 pts.) We want to partially develop and analyze an iterative improvement algorithm for the assignment problem (pp. 116-118). An instance of this problem has n people, n jobs, and an $n \times n$ cost matrix C . Element $C[i, j]$ of this matrix indicates the cost of person i doing job j . The desired output is an assignment of people to jobs so that each person is assigned one job, each job has a person assigned to it, and the total cost is minimized.

We will represent an assignment by a permutation of the numbers from 1 to n . The first number is the job assigned to person 1, the second number is the job assigned to person 2, etc.

- (a) (5 pts.) How many possible assignments are there for an instance with n people and jobs?
- (b) (5 pts.) Show the best and worst assignment for the instance of the assignment problem on p. 116.
- (c) (10 pts.) Given an assignment $A[1..n]$, provide pseudocode that returns true if A is a permutation of the integers from 1 to n . What is the order of growth of your algorithm?
- (d) (10 pts.) Given an assignment $A[1..n]$ and a cost matrix $C[1..n, 1..n]$, provide pseudocode to return the cost of the assignment. What is the order of growth of your algorithm?
- (e) (40 pts.) Let A and B be two different assignments. Create a directed graph where 1 to n are the vertices, and (i_1, i_2) is an edge if $A[i_1] = B[i_2]$, that is, person i_1 's job in A is person i_2 's job in B . This sequence of exercises leads to showing that this graph consists of disjoint cycles or a single cycle.
 - i. Show what this graph looks like for $A = \langle 2, 5, 3, 6, 4, 1 \rangle$ and $B = \langle 6, 4, 2, 1, 5, 3 \rangle$.
 - ii. Show that there is exactly one edge from every vertex, and that there is exactly one edge to every vertex (in general, not just for the above instance). Hint: If person i is doing a job in A (or B), then how many people are doing that job in B (or A)?
 - iii. Show that every vertex is in a cycle (in general, not just for the above instance). Hint: Consider the path starting from any vertex i_1 , i.e., i_1, i_2, i_3, \dots . Supposing i_1 is not in a cycle, then i_2 is not in a cycle (only way to i_2 is from i_1), then i_3 is not in a cycle, \dots Then what?
 - iv. If m vertices are in a cycle, then the cycle has m edges. Can this cycle be connected to any other vertex? Explain. Hint: How many edges can m vertices have?

- (f) (10 pts.) Let A and B be two different assignments, where B costs less than A . Again, create a directed graph where 1 to n are the vertices, and (i_1, i_2) is an edge if $A[i_1] = B[i_2]$. In addition, let the weight on edge (i_1, i_2) be $C[i_2, j] - C[i_1, j]$, where $j = A[i_1] = B[i_2]$. That is, the weight is B 's cost minus A 's cost.

Because the weights include all the individual differences, it follows that the sum of the weights is equal to B 's cost minus A 's cost. If B is better, this is a negative number. Because the graph consists of disjoint cycles, it follows that the sum of the cycles is equal to the sum of the weights. Because the sum of the cycles is negative, then at least one cycle must be a negative cycle.

We can use a negative cycle to improve A 's assignment. For each edge (i_1, i_2) in a negative cycle, assign job $A[i_1]$ to i_2 .

This exercise shows what one instance looks like.

- i. Show what this weighted graph looks like for the assignment instance on p. 116, and $A = \langle 3, 1, 4, 2 \rangle$ and $B = \langle 1, 3, 2, 4 \rangle$
- (g) (20 pts.) Let A be an assignment. Create a complete weighted directed graph where 1 to n are the vertices, and edge (i_1, i_2) has weight $C[i_2, j] - C[i_1, j]$, where $j = A[i_1]$. If A is not optimal, we know this graph contains a negative cycle. We can use the Bellman-Ford algorithm to find a negative cycle, and then use the negative cycle as an augmenting path, i.e., to improve the assignment.
- i. Show what this weighted graph looks like for the assignment instance on p. 116, and $A = \langle 3, 2, 1, 4 \rangle$.
 - ii. Identify a negative cycle in the graph, and specify the new assignment based on your negative cycle.
- (h) (100 pts., shared extra credit) Show how the assignment instance on p. 116 can be specified as a linear programming instance. Alternatively, you can provide a general formulation from any assignment instance to a linear programming instance.

“Shared extra credit” means that a total of 100 points will be allocated to the class. Any mistake will result in no extra credit.