Universal Turing Machines

The Turing machines we have seen so far perform specific computations, but our computers are programmable and general-purpose.

A *universal Turing machine* is a TM that, given a description of a TM M and a string w, can simulate the computation of M on w any TM on any input.

We need to show how any TM and any string using any alphabet can be encoded.

Encoding Turing Machines

Ex. encoding with tape alphabet $\{0, 1, :, \square\}$.

State names in binary, e.g., q_0 through q_{13} as 0000 through 1101.

Symbol names in binary, e.g., a through z as 00000 through 11001.

Use : as a separator for input and transitions.

abbaab can be encoded as 0:1:1:0:0:1.

$\delta(q_2, a) = (q_3, b, L)$ encoded as 0010:0:0011:1:0.

Constructing a Universal TM

Suppose the UTM has three tapes:

1. TM description, an encoded definition of M
2. TM tape, the encoded tape contents of M
3. current state of TM, the encoded internal state of M

It is possible to use a single tape, carefully keeping the 3 parts separated.

The UTM operates by:

1. Find a transition on tape 1 that matches the current state and symbol on tapes 3 and 2.
2. Write the new state on tape 3.
3. Write the symbol given by the transition on tape 2.
4. Move the pointer on tape 2.

Obviously, for a real UTM, many details are missing, but the end result will be a UTM.

Countability of TMs

A set is *countable* if some sequence has all the elements. Here are some countable infinite sets.

Positive Integers: $1, 2, 3, 4, 5, \ldots$

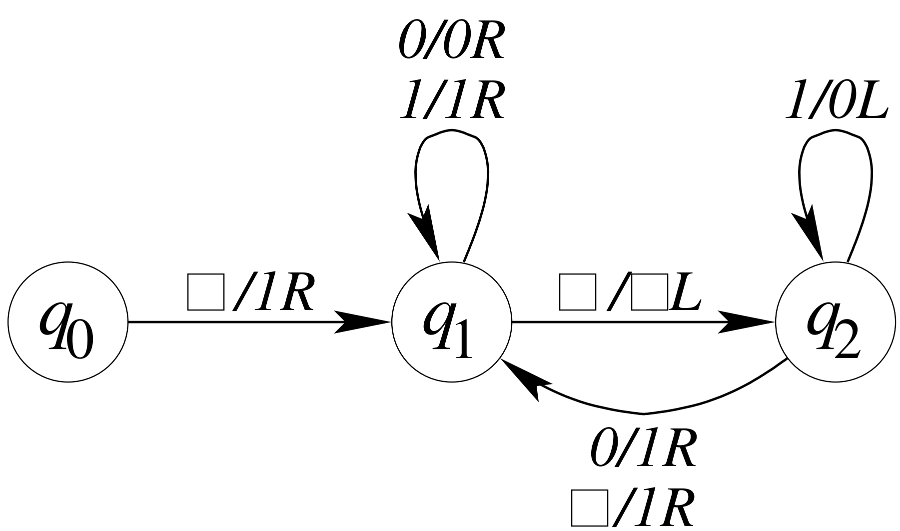
Integers: $0, 1, -1, 2, -2, 3, -3, \ldots$

Positive fractions: $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \ldots$

Strings on $\{a, b\}$: $\lambda, a, b, aa, ab, ba, bb, \ldots$

An *enumeration procedure* is a method for writing a sequence of elements.

This TM enumerates positive integers in binary.



TMs are countable because strings are countable, TMs can be represented by strings, and it is easy to check if a string is a valid TM.

Uncountabilty

The real numbers are not countable. This can be proven using a *diagonalization*.

Let $x_1, x_2, \ldots$ be an infinite sequence of real numbers.

Let $d(x, i)$ = the ith digit in x's fractional part in the decimal expansion of x.

Let y be a real number such that $d(y, i) \neq d(x_i, i)$.

For all x_i in the sequence, $y \neq x_i$.

$$
\begin{aligned}
x_1 &= 0.\ \boxed{9}\ 6\ \ 3\ \ 7\ \ldots \\
x_2 &= 0.\ 9\ \boxed{8}\ 6\ \ 2\ \ldots \\
x_3 &= 0.\ 4\ \ 9\ \boxed{5}\ 5\ \ldots \\
x_4 &= 0.\ 6\ \ 6\ \ 6\ \boxed{7}\ \ldots \\
y &= 0.\ 4\ \ 3\ \ 0\ \ 2\ \ldots
\end{aligned}
$$

This implies that no sequence of real numbers includes all the real numbers.