Universal Approximation

Cybenko-Hornik-Funahashi Theorem:
Let σ be any sigmoidal function.
Let all inputs be in [0 1] or other finite interval.
Let I_d the d-dimensional cube $[0, 1]^d$
Then a sum of the form:

$$f_a(\mathbf{x}) = \sum_j w_j \sigma(b_j + \mathbf{v}_j \cdot \mathbf{x})$$

can approximate any continuous function f to any accuracy. It might require any number of hidden neurons.

Universal Logical Approximation

We demonstrate a weaker result: that any logical expression can be converted to a neural network. However, the number of hidden neurons might be exponential in the number of inputs.

Let inputs be in $\{-1, 1\}$, using -1 for false and 1 for true. A *literal* is an input or its negation, e.g., $x_1, \neg x_1, x_2, \neg x_2, \ldots$

Theorem:
A single neuron can represent a conjunction (AND) of literals or a disjunction (OR) of literals.

Proof:
Use a weight of 2 for positive literals, -2 for negative literals, and 0 otherwise.
For k literals, use bias of $2k-1$ for disjunction, or $1-2k$ for conjunction.
With tanh activation, we can multiply all weights by a large number so that the output will be close to -1 when false or 1 when true.

Theorem:
A training set with p positive examples can be represented by a NN with p hidden neurons.

Proof:
Assign one hidden neuron/example.
A hidden neuron outputs 1 for its example, else -1.
Each hidden neuron does an AND.
The output neuron outputs 1 if any hidden neuron is 1.
The output neuron does an OR.

Disjunctive Normal Form Definition

A *term* is a conjunction (AND) of literals.

A DNF formula is a disjunction (OR) of terms.

Theorem:

DNF can represent any Boolean formula.

Proof:

List all truth assignments that make it true.

Create a term for each truth assignment.

Theorem:

A DNF formula with p terms can be represented by a ANN with p hidden neurons.

Proof:

Create a hidden neuron for each term.

Each hidden neuron performs a conjunction.

Output neuron outputs 1 if any hidden neuron is 1.

The output neuron performs a disjunction.

Conjunctive Normal Form Definition

A *clause* is a disjunction (OR) of literals.

CNF is a conjunction of clauses.

Theorem:

CNF can represent any Boolean formula.

Proof:

List all truth assignments that make it false.

Each clause negates a truth assignment.

Theorem:

A CNF formula with p clauses can be represented by a ANN with p hidden neurons.