Analysis of Iterative Deepening

To talk about iterative deepening and other search algorithms, I will use the following notation and assumptions:

> The search graph is a uniform search tree with the initial state as the root and with branching factor b, $b \geq 2$. By uniform, I mean that each non-leaf state has b children.
>
> m is the depth of the search tree from the initial state. I.e., each state on levels 0 through $m-1$ have b children. Each state on level m has 0 children.
>
> There is a goal state within distance d of the initial state.

b, m, and d are integers. From these assumptions, the following can be demonstrated.

Theorem 1 *For each integer l, $0 \leq l \leq m$, there are b^l states on level l of the search tree.*

Proof: By mathematical induction.

Basis: There is one state on level 0 because level 0 consists only of the initial state. Note that $1 = b^0$.

Assumption: Let l be any integer between 0 and $m-1$. Suppose that level l of the search tree contains b^l states.

Induction: Because b is the branching factor, each state on level l has b children on level $l+1$. Thus, there are $b \cdot b^l = b^{l+1}$ states on level $l+1$.

So, by mathematical induction, the theorem is proved.

End Proof.

Theorem 2 *Let l be any integer between 0 and m. The number of states from level 0 to level l is:*

$$\frac{b^{l+1}-1}{b-1}$$

Proof: The number of states on all the levels from level 0 to level l is:

$$\sum_{i=0}^{l} b^i = \frac{(b-1)\left(\sum_{i=0}^{l} b^i\right)}{b-1} = \frac{\sum_{i=0}^{l}\left(b^{i+1}-b^i\right)}{b-1} = \frac{b^{l+1}-1}{b-1}$$

End Proof.

Corollary 3 *The number of states visited by depth-first search is at most:*

$$\frac{b^{m+1}-1}{b-1}$$

Corollary 4 *The number of states visited by breadth-first search is at most:*

$$\frac{b^{d+1}-1}{b-1}$$

Corollary 5 *The number of states visited by depth-first search with depth limit l is at most:*

$$\frac{b^{l+1}-1}{b-1}$$

Theorem 6 *The number of states visited by iterative deepening is at most:*

$$\frac{b^{d+2}}{(b-1)(b-1)}$$

Proof: Iterative deepening performs $d+1$ depth-limited depth-first searches, from depth limit 0 to depth limit d, so the number of state visits is at most:

$$\sum_{l=0}^{d} \frac{b^{l+1}-1}{b-1} < \sum_{l=0}^{d} \frac{b^{l+1}}{b-1} = \frac{\sum_{l=0}^{d} b^{l+1}}{b-1}$$

Considering the numerator:

$$\sum_{l=0}^{d} b^{l+1} = \sum_{l=1}^{d+1} b^{l} < \sum_{l=0}^{d+1} b^{l} = \frac{b^{d+2}-1}{b-1} < \frac{b^{d+2}}{b-1}$$

Thus,

$$\sum_{l=0}^{d} \frac{b^{l+1}-1}{b-1} < \frac{\sum_{l=0}^{d} b^{l+1}}{b-1} < \frac{b^{d+2}}{(b-1)(b-1)}$$

which proves the theorem.
End Proof.

Note that the ratio of the iterative deepening bound over the breadth-first search bound is approximately $b/(b-1)$, which means that iterative deepening does not search much more than breadth-first search, while using much less memory. Iterative deepening requires memory for $O(bd)$ states, while breadth-first search requires memory for $O(b^d)$ states.

The weakness in the above analysis is the assumption that the search graph is a tree. What will be the result if there are significant numbers of repeated states over different branches of the search graph?