Analysis of Propositional Logic

To talk about propositional logic, I will use the following assumptions and notation:

A theory $F$ is a set of propositional logic formulas.

Each formula $f \in F$ is a set of literals. A literal is a propositional variable or its negations (e.g., $a$ and $\neg a$ are literals). Any formula that has a literal and its negation is a tautology and can be deleted from $F$.

Each formula $f \in F$ is interpreted as a disjunction (“or”), and $F$ is interpreted as a conjunction of the formulas. This is called conjunctive normal form (CNF).

A value assignment $v$ assigns a truth value to each propositional variable used in $F$. It is often useful to think of $v$ as a conjunction of literals.

$v$ satisfies $F$ (equivalently $v$ is a model of $F$, or $v \models F$) if and only if every formula in $F$ has one literal which is true in $v$.

For two formulas $f_1$ and $f_2$, the resolution inference rule works as follows. Let $a$ be a propositional variable. If $a \in f_1$ and $\neg a \in f_2$, then we can infer $f_3 = (f_1 - \{a\}) \cup (f_2 - \{\neg a\})$.

**Theorem 1** The resolution inference rule is sound.

This is easily provable using truth tables.

**Theorem 2** Any arbitrary propositional formula $F'$ can be converted to conjunctive normal form.

**Algorithm:** The conversion algorithm is as follows. Let $F'$ be an arbitrary propositional formula. Then $\text{convert}(F')$ proceeds recursively as follows:

- If $F'$ is a literal or a disjunction of literals, put the literal(s) into a set $f$ and return $\{f\}$.
- Else if $F'$ has the form $\neg\neg F'_1$, then return $\text{convert}(F'_1)$.
- Else if $F'$ has the form $\neg(F'_1 \lor F'_2)$, then return $\text{convert}(\neg F'_1 \land \neg F'_2)$.
- Else if $F'$ has the form $\neg(F'_1 \land F'_2)$, then return $\text{convert}(\neg F'_1 \lor \neg F'_2)$.
- Else if $F'$ has the form $F'_1 \land F'_2$, then return $\text{convert}(F'_1) \cup \text{convert}(F'_2)$.
- Else if $F'$ has the form $F'_1 \lor F'_2$, then let $F_1$ be $\text{convert}(F'_1)$ and let $F_2$ be $\text{convert}(F'_2)$. Now create a new propositional variable $a$. Add $a$ to each formula in $F_1$ and add $\neg a$ to each formula in $F_2$ and return $F_1 \cup F_2$.

**End Algorithm.**
Let $F = \text{convert}(F')$. The claim is that $v' \models F'$ if and only if there is a value assignment $v$ such that $v \models v'$ and $v \models F$, i.e., $v$ assigns appropriate values to the extra variables that are added.

The last step in the algorithm is probably the most mysterious. Suppose a value assignment $v \models F_1 \lor F_2$ (before we create $a$). Now see what happens after we create $a$. If $v \models F_1$, we can choose false for $a$ to make $F_2$ true. If $v \models F_2$, then we can choose true for $a$ to make $F_1$ true, so we really still have $F_1 \lor F_2$. Alternatively, we could have used a distribution law and not add any additional variables, but this could get combinatorially explosive.

So depending on our choice for $a$, we automatically get either $F_1$ or $F_2$.

**Theorem 3** (completeness) *If a propositional theory $F$ in conjunctive normal form is inconsistent, then the resolution rule can be used to infer inconsistent literals.*

**Proof:** We prove this by induction on the number of propositional variables.

**Basis:** When a theory $F$ contains only one propositional variable $a$, then $F$ is inconsistent if and only if both $\{a\} \in F$ and $\{\neg a\} \in F$.

**Induction:** Suppose $F$ contains $n + 1$ propositional variables. If $F$ is inconsistent, then every value assignment $v$ for the variables makes $F$ false, i.e., $v \not\models F$.

Let $a$ be a propositional variable used in $F$. If both $\{a\} \in F$ and $\{\neg a\} \in F$, then the inconsistency is obvious, and no more work is needed.

Otherwise, create a new theory $F'$ as follows. Remove from $F$ any formula that contains both $a$ and $\neg a$ (these formulas are tautologies, anyway). Let $F_a$ be the subset of formulas that contain $a$, let $F_{\neg a}$ be the subset of formulas that contain $\neg a$, and let $F_{\text{other}}$ be the rest of the formulas. Let $F_{\text{new}}$ be the result of resolving $a$ from every pair of formulas $(f_a, f_{\neg a}) \in F_a \times F_{\neg a}$. Let $F' = F_{\text{new}} \cup F_{\text{other}}$. $F'$ does not use $a$, so it has $n$ variables (or less).

Suppose $F$ is consistent. Then $v \models F$ for some value assignment $v$. Because the resolution inference rule is sound, then it must be the case that $v \models F'$.

Suppose $F$ is inconsistent. We show that $F'$ is also inconsistent. Let $v'$ be an arbitrary value assignment for $F'$. Let $v_a$ be identical to $v'$ except that $v_a$ assigns true to $a$. Let $v_{\neg a}$ be identical to $v'$ except that $v_{\neg a}$ assigns false to $a$. Because $F$ is inconsistent, it must be the case that $v_a \not\models F_a$ or $v_{\neg a} \not\models F_{\neg a}$ or $v_a \not\models F_{\text{other}}$. Also, it must be the case that $v_{\neg a} \not\models F_a$ or $v_{\neg a} \not\models F_{\text{other}}$. Consider the following cases:

**Case 1:** $v_a \not\models F_a$ or $v_{\neg a} \not\models F_{\neg a}$. Neither of these are possible because, respectively, assigning true to $a$ makes $F_a$ true, and assigning false to $a$ makes $F_{\neg a}$ true.

**Case 2:** $v_a \not\models F_{\text{other}}$ or $v_{\neg a} \not\models F_{\text{other}}$. This implies that $v' \not\models F_{\text{other}}$ because $a$ is not referenced by $F_{\text{other}}$. Note that $v' \not\models F_{\text{other}}$ implies $v' \not\models F'$ because $F_{\text{other}}$ is a subset of $F'$.

**Case 3:** $v_a \not\models F_{\neg a}$ and $v_{\neg a} \not\models F_a$. First, $v_a \not\models F_{\neg a}$ implies that all the literals in some formula $f' \in F_{\neg a}$ are made false by $v_a$. Second, $v_{\neg a} \not\models F_a$ implies that all the literals in some formula $f \in F_a$ are made false by $v_{\neg a}$. When resolution is applied to $(f, f')$, $a$ is eliminated, so it must be that both $v_a$ nor $v_{\neg a}$ makes the new formula false, which implies that $v'$ makes the new formula false. Because the new formula is in $F'$, it follows that $v' \not\models F'$.

So by mathematical induction, if $F$ is inconsistent, then the resolution rule is able to infer inconsistent literals.

**End Proof.**