

Problem Set 1

CS 5633 – Spring 2007
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assigned January 22, 2007
first due date February January 31, 2007
second due date February 7, 2007

Your solutions must be submitted as a document to WebCT.

1. What are the loop invariants of the outer and inner loops of the algorithm below?

```
SELECTION-SORT(A)
  n ← length(A)
  for i ← 1 to n - 1 do
    min ← i
    for j ← i + 1 to n do
      if A[j] < A[min] then min ← j
    exchange A[i] ↔ A[min]
```

2. Suppose n is a positive integer that we wish to encode to within 0.1% accuracy? Asymptotically bound the number of bits that are needed. Justify your answer.
3. Provide the simplest function $g(n)$ such that $\sum_{i=1}^n (n - i)/i$ is $\Theta(g(n))$.
4. Rank the following 12 functions by order of growth. Note any pairs of functions such that $f(n)$ is $\Theta(g(n))$.

$$f_1(n) = \lg n$$

$$f_2(n) = n^2$$

$$f_3(n) = 2^n$$

All the functions $f_i(f_j(n))$ for $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3\}$

5. Use mathematical induction to show that when n is an exact power of 2, the solution of the recurrence

$$T(n) = \begin{cases} a & \text{if } n = 1, \\ 2T(n/2) + n & \text{if } n = 2^k, \text{ for some } k > 0 \end{cases}$$

is $T(n) = n(a + \lg n)$.

6. Consider recurrences of the form $T(n) = aT(n/b) + n^c$. Fill in the following table with asymptotic bounds in Θ -notation.

a	b	c	Asymptotic Bound
1	2	0	
1	2	1	
1	2	2	
2	2	0	
2	2	1	
2	2	2	
3	2	0	
3	2	1	
3	2	2	

7. The following four exercises are concerned with the number of changes to a variable in an algorithm that keeps track of the current median.
- Define a median of n numbers to be any number m such that $\lceil n/2 \rceil$ numbers are less than or equal to m and $\lfloor n/2 \rfloor$ numbers are greater than or equal to m . Using this definition, provide 5 different medians of the numbers 1, 2, 3, 4, 5, 6.
 - In pseudocode, write an algorithm that finds the median of an array of n numbers by a loop that keeps track of the median of the first i numbers. Do not change the value of the current median if it satisfies the above definition. For example, if the array is 1, 5, 6, 2, 3, 4, the value of the current median should be the sequence 1, 1, 5, 5, 3, 3.
 - What is the minimum and maximum number of changes to the current median for an array of size n ? Provide examples illustrating both cases.
 - In this exercise, we want to prove that the expected number of changes is greater than $n/4$. Suppose that the n numbers in the array are distinct and in random order. Suppose m_i is the median of the first i numbers where i is odd. Derive a bound on the probability that $m_{i+2} \neq m_i$. Hint: The next two numbers might be both lower than m_i or both higher than m_i .