The Minimum and the Maximum

Min-and-Max(A)

$n \leftarrow length[A]$
$min \leftarrow A[n]$
$max \leftarrow A[n]$
for $i \leftarrow 1$ **to** $n-1$ **by** 2 **do**
  if $A[i] < A[i+1]$ **then**
    if $A[i] < min$ **then** $min \leftarrow A[i]$
    if $A[i+1] > max$ **then** $max \leftarrow A[i+1]$
  else
    if $A[i+1] < min$ **then** $min \leftarrow A[i+1]$
    if $A[i] > max$ **then** $max \leftarrow A[i]$

---

3 comparisons are made for every pair of elts.
$3n/2$ comparisons are made when n is even.
Why is the algorithm correct?
What if n is odd?

Order Statistics

The ith order statistic is the ith smallest elt.

R-Select(A, p, r, i)
  if $p = r$ **then return** $A[p]$
  $q \leftarrow$ R-Partition(A, p, r)
  $k \leftarrow q - p + 1$
  if $i = k$ **then return** $A[q]$
  else if $i \leq k$
   then return R-Select$(A, p, q-1, i)$
  else return R-Select$(A, q+1, r, i-k)$

Worst-case: $T(n) = T(n-1) + n - 1 \in \Theta(n^2)$

Best-case: $T(n) = T(i) + n - 1$, followed by $T(i) = T(1) + i - 1$, which is $\Theta(n)$.

Average region is approximately $2n/3$

Assume $p = 1$ and $r = n$,
i is equally likely to be any value from 1 to n,
q is equally likely to be any value from 1 to n.

Condition	Probability	Region Size
$i = q$	$1/n$	1
$i < q$	$(q-1)/n$	$q-1$
$i > q$	$(n-q)/n$	$n-q$

Average region

$$= \frac{1}{n} \sum_{q=1}^{n} \left(\frac{1}{n}(1) + \frac{q-1}{n}(q-1) + \frac{n-q}{n}(n-q) \right)$$

$$= \frac{1}{n} \sum_{q=1}^{n} \frac{1 + (q^2 - 2q + 1) + (n^2 - 2nq + q^2)}{n}$$

$$= \sum_{q=1}^{n} \left(\frac{n^2 - 2nq - 2q + 2q^2 + 2}{n^2} \right)$$

$$= \frac{n^3 + 2n}{n^2} - \frac{2(n+1)n(n+1)}{2n^2} + \frac{2n(n+1)(2n+1)}{6n^2}$$

$$= \frac{2n^2 - 3n + 4}{3n}$$

$$< \frac{2n}{3} \text{ when } n \geq 2$$

Note that $T(n) = T(2n/3) + n - 1$ is $\Theta(n)$.

Average-Case by Mathematical Induction:
I will use a bound on the recurrence:

$T(1) = 0$

$$T(n) \leq n - 1 + \frac{1}{n} \sum_{q=1}^{n} T(\max(1, q-1, n-q))$$

Basis: $T(1) = 0$, so $T(n) \leq cn$ for $n = 1$.

Assume: $T(q) \leq cq$ for $1 \leq q \leq n-1$.

Show: $T(n) \leq cn$

$$\begin{aligned} & T(n) \\ & \leq n - 1 + \frac{1}{n} \sum_{q=1}^{n} T(\max(1, q-1, n-q)) \\ & \leq n - 1 + \frac{1}{n} \sum_{q=1}^{n} c \max(1, q-1, n-q) \\ & \leq n - 1 + \frac{1}{n}\, n\, c \frac{3n}{4} \\ & \leq n + \frac{3\, c\, n}{4} \\ & \leq c\, n \quad \text{when } c \geq 4 \end{aligned}$$

$\Theta(n)$ Worst-Case Selection

Select(A, p, r, i)
  if $p = r$ **then return** $A[p]$
  $x \leftarrow$ Good-Pivot(A, p, r)
  $q \leftarrow$ Partition-With-Pivot(A, p, r, x)
  $k \leftarrow q - p + 1$
  if $i = k$ **then return** $A[q]$
  else if $i \leq k$
   then return Select$(A, p, q - 1, i)$
  else return Select$(A, q + 1, r, i - k)$

Good-Pivot(A, p, r)
  for $j \leftarrow 0$ **to** $\lfloor (r - p)/5 \rfloor$ **do**
    $B[j] \leftarrow$ median of $A[p + 5j]$
            thru $A[\min(p + 5j + 4, r)]$
  return
    Select$(B, 0, \lfloor (r - p)/5 \rfloor, \lfloor (r - p)/10 \rfloor)$

In the worst-case, Select is recursively called on regions of size $n/5$ and $7n/10$.