

Shortest Path Problems on a Polyhedral Surface*

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Abstract

We develop algorithms to compute shortest path edge sequences, Voronoi diagrams, the Fréchet distance, and the diameter for a polyhedral surface.

1 Introduction

Two questions are invariably encountered when dealing with shortest path problems. The first question is how to represent the combinatorial structure of a shortest path. In the plane with polygonal obstacles, all shortest path vertices occur at obstacle vertices, so a shortest path can be combinatorially described as a sequence of obstacle vertices [8]. On a polyhedral surface, a shortest path need not turn at vertices [11], so a shortest path is often described combinatorially by an *edge sequence* that represents the sequence of edges encountered by the path [1].

The second commonly encountered shortest path question is how to compute shortest paths. Let M be the complexity of a polyhedral surface. On a *convex* polyhedral surface, Mount [12] shows that $\Theta(M^4)$ combinatorially distinct shortest paths exist, and Schevon and O’Rourke [13] show that only $\Theta(M^3)$ of these shortest paths are *maximal* (such paths cannot be extended at either end without creating a sub-optimal path). Agarwal et al. [1] use these properties to compute $\Theta(M^4)$ shortest path edge sequences in $O(M^6 2^{\alpha(M)} \log M)$ time, where $\alpha(M)$ is the inverse Ackermann function. They also compute the diameter of a convex polyhedral surface in $O(M^8 \log M)$ time.

We improve the diameter and edge sequence algorithms of Agarwal et al. [1] on a convex polyhedral surface by a linear factor. Our improvement combines star unfolding and kinetic Voronoi diagram techniques. In addition, we achieve a linear factor improvement over the Fréchet distance algorithm of Maheshwari and Yi [10] for polygonal curves on a convex polyhedral surface, and we present the first algorithm to compute the Fréchet distance between polygonal curves on a non-convex polyhedral surface. Our motivation for studying the Fréchet distance on a polyhedral surface is that teaming up two people for

safety reasons is common practice in many real-life situations, ranging from scouts in summer camp, to fire fighters and police officers, and even to astronauts exploring the moon. In all of these applications, two team members need to coordinate their movement in order to stay within “walking distance” so that fast assistance can be offered in case of an emergency. The Fréchet distance is an ideal model for this scenario. We summarize our results in Table 1. The following notation is used throughout this paper. Let M be the total complexity of a polyhedral surface and two polygonal curves A and B that lie on the surface. Let $\pi(s, t)$ denote a shortest path on a surface between points s and t , and let $d(s, t)$ be the Euclidean length of $\pi(s, t)$.

2 Shortest Path Problems on a Polyhedral Surface

2.1 Shortest Path Edge Sequences

This section contains superset and exact algorithms for computing the $\Theta(M^4)$ shortest path edge sequences on a convex polyhedral surface \mathcal{P} . Both of our algorithms are linear factor improvements over related results of Agarwal et al. [1]. Let v_1, \dots, v_M be the corner vertices of \mathcal{P} , and let $\Pi = \{\pi(s, v_1), \dots, \pi(s, v_M)\}$ be an angularly ordered set of non-crossing shortest paths from a source point $s \in \mathcal{P}$ to each corner vertex $v_j \in \mathcal{P}$. The *star unfolding* \mathcal{S} is a simple polygon [4] defined by cutting \mathcal{P} along each of the shortest paths in Π and unfolding the resulting shape into the plane. Since the source point s touches all M cuts, $s \in \mathcal{P}$ maps to M image points s_1, \dots, s_M on the (two dimensional) boundary of the unfolded simple polygon \mathcal{S} (see Figure 1).

An *equator* [5] in the star unfolding is a closed polygonal curve through the points v_1, \dots, v_M, v_1 . The region inside the equator contains no source image and is called the *core* [7]. The disconnected regions outside the core each contain a source image and are collectively referred to as the *anti-core* [7]. A *core edge* is an image of an edge of \mathcal{P} that was never cut during the unfolding process. Each of the $O(M)$ core edges has both of its endpoints at vertices of \mathcal{S} and is entirely contained in the core (see Figure 1b). Each of the $\Theta(M^2)$ *anti-core edges* is an image of an edge of \mathcal{P} that was cut during the unfolding process.

The star unfolding of s can be used to compute $\pi(s, t)$ for points $s, t \in \mathcal{P}$ as follows. The shortest

*This work has been supported by the National Science Foundation grant NSF CAREER CCF-0643597.

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	Time	Space
Star Unfolding maintained over all edges	$O(M^4)^*$	$O(M^4)$
Kinetic Voronoi Diagram maintained over all edges	$O(M^5 2^{\alpha(M)} \log M)$	$O(M^5 2^{\alpha(M)})$
Edge Sequences (superset)	$O(M^5)$	$O(M^5)$
Edge Sequences (exact)	$O(M^5 2^{\alpha(M)} \log M)$	$O(M^4 2^{\alpha(M)})$
Diameter	$O(M^7 \log M)$	$O(M^4)$
Fréchet	$O(M^6 \log^2 M)$	$O(M^2)$
Distance	$O(M^7 \log^2 M)^*$	$O(M^3)$

Table 1: Summary of our contributions. An asterisk indicates a result for a *non-convex* polyhedral surface.

path $\pi(s, t)$ always originates from one of the source images s_1, \dots, s_M . If the image of t lies in an anti-core region containing s_i , then the optimal shortest path must originate from s_i (i.e., the shortest paths from t to the remaining source images are suboptimal). If the image of t lies in the core, then the nearest source image is determined with Voronoi diagram techniques.

Agarwal et al. [1] partition the M edges of the convex polyhedral surface \mathcal{P} into $O(M^3)$ line segment *edgelets* such that all source points on an edgelet can be associated with the same combinatorial star unfolding. For each edgelet, a star unfolding is computed, and $O(M^3)$ edge sequences are extracted from it. This yields an $O(M^6)$ superset of the $\Theta(M^4)$ shortest path edge sequences for \mathcal{P} in $O(M^6)$ time and space [1]. Agarwal et al. [1] also show how to compute the exact set of $\Theta(M^4)$ shortest path edge sequences in $O(M^6 2^{\alpha(M)} \log M)$ time. We speed up both of these algorithms by a linear factor.

Theorem 1 *A star unfolding can be maintained as a source point s varies continuously over all M edges of a (possibly non-convex) polyhedral surface in $O(M^4)$ time and space.*

Proof Sketch. The set Π of shortest paths to each corner vertex defines a combinatorial star unfolding. These paths can only change at edgelet endpoints, so Π can be maintained over a discrete set of events. For each change to Π , we update $O(1)$ anti-core regions and possibly all $O(M)$ core edges in the star unfolding. \square

Theorem 2 *A superset of the $\Theta(M^4)$ shortest path edge sequences for a convex polyhedral surface \mathcal{P} can be constructed in $O(M^5)$ time and space.*

Proof. Each edgelet defines a star unfolding with source images s_1, \dots, s_M . For each s_i , construct an edge sequence from s_i to each of the $O(M)$ anti-core edges in the anti-core region for s_i and to each of

the $O(M)$ core edges. This yields $O(M^2)$ edge sequences per edgelet, and $O(M^5)$ edge sequences over all edgelets. This is sufficient to determine the desired superset because only core edges have shortest path edge sequences to multiple sites, and this approach considers all possibilities. \square

The *exact* set of $\Theta(M^4)$ shortest path edge sequences can be determined with kinetic Voronoi diagram techniques. Albers et al. [2] show that for point sites moving along line segments at constant speeds, each *pair* of sites defines $O(M 2^{\alpha(M)})$ events, and each event is handled in $O(\log M)$ time.

Theorem 3 *A kinetic Voronoi diagram of source images can be maintained in $O(M^4 2^{\alpha(M)} \log M)$ time and $O(M^4 2^{\alpha(M)})$ space as a source point varies continuously over one edge e of a convex polyhedral surface \mathcal{P} .*

Proof. A kinetic Voronoi diagram for the *first* edgelet on e has $O(M^2 \cdot M 2^{\alpha(M)})$ events [2] due to the linear motion of $O(M^2)$ *pairs* of source images in the star unfolding. Each of the $O(M^2)$ subsequent edgelets on e can be handled by removing one site (i.e., a source image) and adding a new site. All other sites continue to be parameterized along the same line segments as in the previous edgelet. Thus, each of the $O(M^2)$ edgelets contributes $M - 1$ new *pairs* of sites and adds $O(M \cdot M 2^{\alpha(M)})$ new events to the event queue. \square

We construct the exact set of shortest path edge sequences for an edgelet α as follows. Maintaining a kinetic Voronoi diagram for α yields a two-dimensional parameterized Voronoi cell φ_i for each source image s_i . The unique edge sequence in the star unfolding's dual graph from s_i to a core edge e represents a shortest path if and only if e intersects φ_i for some $s \in \alpha$.

To decide in logarithmic time whether e intersects φ_i , Agarwal et al. [1] represent each parameterized

Voronoi vertex as an algebraic curve. For a vertex $v_i \in \mathcal{S}$ that touches the anti-core region for the source image s_i , they triangulate the region of the core that is directly visible to v_i such that every triangle Δ has apex v_i . Using polar coordinates centered at v_i , they compute an upper envelope μ of the algebraic curves defining φ_i and partition these curves such that each curve segment is fully contained in one of these triangles. Inside each triangle Δ , the dual graph of the core has at most one degree three vertex because each core edge is a *chord* of Δ . Agarwal et al. [1] compute a dual graph *tree* D_Δ for each triangle in $O(M)$ time.

Lemma 4 D_Δ can be constructed implicitly in $O(\log M)$ time.

Proof Sketch. Determine the portion of the core’s dual graph D that lies inside Δ by point locating the three vertices of Δ in D . \square

For each constant complexity algebraic curve on the upper envelope $\mu \cap \Delta$, Agarwal et al. [1] perform a binary search on the two paths in D_Δ . The deepest edge on each of these two paths that intersects some arc of $\mu \cap \Delta$ defines a *maximal* shortest path edge sequence. The set of all prefixes of these maximal sequences defines $\Theta(M^4)$ shortest path edge sequences.

Theorem 5 The $\Theta(M^4)$ shortest path edge sequences for a convex polyhedral surface \mathcal{P} with M edges can be constructed in $O(M^5 2^{\alpha(M)} \log M)$ time and $O(M^4 2^{\alpha(M)})$ space.

Proof. Let n_i be the total number of parameterized Voronoi vertices over all edgelets, and let t_Δ be the time to process each triangle Δ . The technique of Agarwal et al. [1] requires $O(n_i \log M + M^5 t_\Delta)$ time. Since they assume $n_i \in O(M^6 2^{\alpha(M)})$ and $t_\Delta \in O(M)$, they obtain $O(M^6 2^{\alpha(M)} \log M)$ time. By Theorem 3, $n_i \in O(M^5 2^{\alpha(M)})$ over all $O(M)$ edges of \mathcal{P} , and by Lemma 4, $t_\Delta \in O(\log M)$ time. \square

2.2 Diameter

The *diameter* of a convex polyhedral surface is the largest shortest path distance between any pair of points on the surface. Agarwal et al. [1] compute the diameter in $O(M^8 \log M)$ time by defining $O(M^4)$ *ridge-free regions* on the surface such that all source points in a ridge-free region have the same combinatorial star unfolding.

Theorem 6 The diameter of a convex polyhedral surface \mathcal{P} with M edges can be constructed in $O(M^7 \log M)$ time and $O(M^4)$ space.

Proof Sketch. Maintain a kinetic Voronoi diagram over all ridge-free regions. Each of the $O(M^4)$ ridge

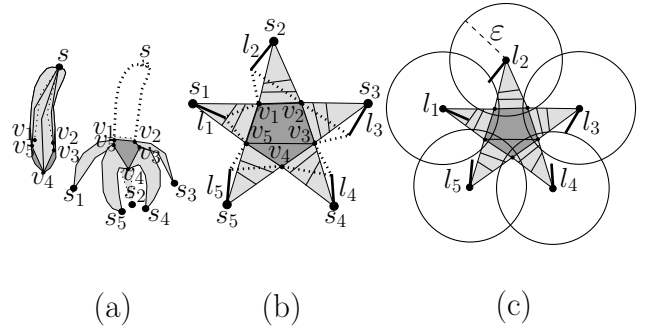


Figure 1: The star unfolding of a polyhedral surface.

free regions defines $O(M^3)$ new events, for a total of $O(M^7)$ parameterized Voronoi vertices. Each vertex can be associated with a function that defines the distance from this vertex to its defining source image. The maximum value defined by these functions is the diameter. \square

2.3 Fréchet Distance

The *Fréchet distance* [3] is a similarity metric for *continuous* shapes that is defined for two polygonal curves $A, B : [0, 1] \rightarrow \mathbb{R}^d$ as

$$\delta_F(A, B) = \inf_{\alpha, \beta: [0, 1] \rightarrow [0, 1]} \sup_{t \in [0, 1]} d(A(\alpha(t)), B(\beta(t)))$$

where α and β range over continuous non-decreasing reparameterizations, and d is a distance metric for points. For a given constant $\varepsilon \geq 0$, *free space* is defined as $\{(s, t) \mid s \in A, t \in B, d(s, t) \leq \varepsilon\}$. Let $\delta_C(A, B)$ (resp. $\delta_N(A, B)$) denote the Fréchet distance between polygonal curves A and B on a convex (resp. non-convex) polyhedral surface. Maheshwari and Yi [10] have previously shown how to compute $\delta_C(A, B)$ in $O(M^7 \log M)$ time by enumerating all edge sequences. However, their approach relies on [9] whose key claim “has yet to be convincingly established” [1]. We compute $\delta_C(A, B)$ without enumerating edge sequences in $O(M^6 \log^2 M)$ time and $O(M^2)$ space.

We first describe all free space for a given constant $\varepsilon \geq 0$. The star unfolding \mathcal{S} maps a fixed source point $s \in A$ to a set of source image points s_1, \dots, s_M and maps B to a set of core and anti-core edges. *Free space* is defined by the union of a set of disks d_1, \dots, d_M , where each disk d_i has radius ε and is centered at s_i . As the source point s varies continuously on an edgelet, the core is fixed, and each s_i is parameterized along a line segment l_i in the star unfolding (see Figure 1b).

Theorem 7 $\delta_C(A, B)$ can be computed in $O(M^6 \log^2 M)$ time and $O(M^2)$ space.

Proof Sketch. A star unfolding is maintained over all edgelets by Theorem 1. The free space for an *anti-core* edge is defined by the intersection of this edge with one nearest disk d_i . The free space for each *core* edge is defined by the intersection of this edge with $d_1 \cup \dots \cup d_M$. After computing all free space, reachability information [3] can be propagated through the free space via plane sweep [6], and parametric search [3] can be applied to a set of algebraic free space vertex curves. \square

For a *non-convex* polyhedral surface, the star unfolding can overlap itself [5] but is still defined by a set of shortest paths from the source to every corner vertex [5, 11]. Thus, a core can still be defined by a polygonal equator with $O(M)$ complexity. Since shortest paths only turn at vertices in the star unfolding [11], an anti-core region containing s_i must have an *hourglass* shape [8, 6] because it is bounded by two line segments and two shortest paths in the plane.

Theorem 8 $\delta_N(A, B)$ can be computed in $O(M^7 \log^2 M)$ time and $O(M^3)$ space.

Proof Sketch. Maintain a star unfolding over all edgelets. One hourglass defines the free space to an anti-core edge, and $O(M)$ hourglasses define the free space to a core edge. After computing all free space, a combination of plane sweep and parametric search techniques yields $\delta_N(A, B)$. \square

3 Conclusion

We develop algorithms to compute edge sequences, Voronoi diagrams, the Fréchet distance, and the diameter for a polyhedral surface. Our work speeds up the edge sequence and diameter approaches of Agarwal et al. [1] by a linear factor and introduces many algorithms that apply to convex and non-convex polyhedral surfaces. Future work could attempt to compute the $\Theta(M^4)$ shortest path edge sequences in $\Theta(M^4)$ time. Additional shortest path map and link distance results were omitted from this version due to space concerns.

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