Comparison of Distance Measures for Planar Curves

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Abstract

The *Hausdorff distance* is a very natural and straightforward distance measure for comparing geometric shapes like curves or other compact sets. Unfortunately, it is not an appropriate distance measure in some cases. For this reason, the *Fréchet distance* has been investigated for measuring the resemblance of geometric shapes which avoids the drawbacks of the Hausdorff distance. Unfortunately, it is much harder to compute. Here we investigate under which conditions the two distance measures approximately coincide, i.e. the pathological cases for the Hausdorff distance cannot occur. We show that for closed convex curves both distance measures are the same. Furthermore, they are within a constant factor of each other for so-called κ-straight curves, i.e., curves where the arc length between any two points on the curve is at most a constant κ times their Euclidean distance. Therefore, algorithms for computing the Hausdorff distance can be used in these cases to get exact or approximate computations of the Fréchet distance, as well.

1 Introduction

The task of comparing two two-dimensional shapes arises naturally in many applications, e.g., in computer graphics, computer vision and computer aided design. Often two-dimensional shapes are given by the planar curves forming their boundaries. We consider curves as being represented by *parametrizations*, i.e., continuous mappings $f: [0,1] \to \mathbb{R}^2$.

So we are faced with the problem of comparing the 'similarity' of two planar curves. There are several possible distance measures to assess the resemblance of two curves P and Q; we will consider the *Fréchet distance* $\delta_F(P,Q)$, the *weak Fréchet distance* $\tilde{\delta}_F(P,Q)$, and the *Hausdorff distance* $\delta_H(P,Q)$.

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Definition 1. *Let $P, Q\colon [0,1] \to \mathbb{R}^2$ be curves, and $||.||$ denote the L_2 norm.*

- *Identifying P and Q with the sets of points lying on these curves $\delta_H(P,Q)$ denotes the* Hausdorff distance *between P and Q, defined as*

$$\delta_H(P,Q) = \max\left(\tilde{\delta}_H(P,Q), \tilde{\delta}_H(Q,P)\right), \text{ where}$$
$$\tilde{\delta}_H(P,Q) = \sup_{x \in P} \inf_{y \in Q} ||x-y||,$$

  is called the directed Hausdorff distance *from P to Q, and*

- *$\delta_F(P,Q)$ denotes the* Fréchet distance *between P and Q, defined as*

$$\delta_F(P,Q) := \inf_{\rho,\sigma} \max_{t \in [0,1]} ||P(\rho(t)) - Q(\sigma(t))||,$$

  where $\rho, \sigma : [0,1] \to [0,1]$ range over all continuous and increasing functions with $\rho(0) = \sigma(0) = 0$ and $\rho(1) = \sigma(1) = 1$.

  If we do not require ρ and σ to be increasing, we obtain a distance measure that is called the weak Fréchet distance *between P and Q, and is denoted by $\tilde{\delta}_F(P,Q)$.*

The Hausdorff distance between two curves P and Q is the smallest δ, such that P is completely contained in the δ-neighborhood of Q, and vice versa. It is somehow the most straightforward and natural distance measure between curves or other compact sets. But since it does not consider the course of curves there are examples where it is not appropriate in the sense that it is rather small for curves which do not resemble each other at all(see Figure 1). The reason is that the mapping which assigns to each point on P its closest point on Q is not continuous. The Fréchet distance deals with this problem. As a popular illustration of it imagine a man is walking his dog, he is walking on one curve, the dog on the other. Both are allowed to control their speed but are not allowed to go backwards. Then the Fréchet distance of the curves is the minimal length of a leash that is necessary. If both are also allowed to go backwards, we get the weak Fréchet distance between the two curves.

Obviously, for any two curves P and Q the following inequality holds:

$$\delta_H(P,Q) \leq \tilde{\delta}_F(P,Q) \leq \delta_F(P,Q),$$

but as shown in Figures 1 and 2 neither the ratio between δ_H and δ_F, nor the ratio between δ_F and $\tilde{\delta}_F$ is bounded in general.

For given polygonal curves P, Q with n and m vertices, respectively, one can compute $\delta_H(P,Q)$ in $\mathcal{O}\big((m+n)\log(m+n)\big)$ time, see [2], and $\delta_F(P,Q)$ as well as $\tilde{\delta}_F(P,Q)$in why not $\mathcal{O}(mn\log(mn))$] time,see [4].

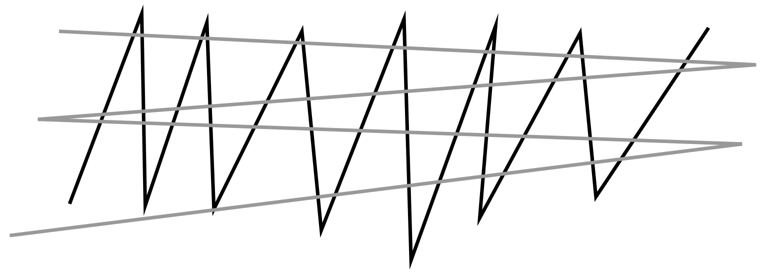

Figure 1: Two curves with small Hausdorff distance having a large Fréchet distance.

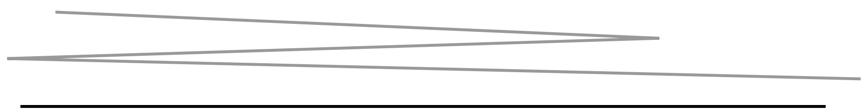

Figure 2: Two curves with small weak Fréchet distance having a large Fréchet distance.

So the Fréchet distance avoids the pitfalls of the Hausdorff distance as in Figure 1 but, not surprisingly, is much harder to compute. In this paper we will consider classes of curves which are less "wiggly" and, therefore, not as pathological for the Hausdorff distance as the ones in Figure 1. In fact, we will see that for closed convex curves both distance measures are even identical. For κ-straight curves, i.e., curves where the arc length between any two points on the curve is at most a constant κ times their Euclidean distance the Fréchet distance is within a constant factor of the Hausdorff distance. Therefore, in these cases more efficient algorithms for computing or, at least, approximating the Fréchet distance can be found.

2 Closed Convex Curves

The result of this section is that Hausdorff and Fréchet distance are the same for convex closed curves. A preliminary version of this result was presented in [3]. In [8] it was shown that it can be applied to convex surfaces in higher dimensions, as well.

We define a *simple closed curve* as a subset of $\mathbb{R}^2$ that is homeomorphic to the circle S^1. We will assume that simple closed curves are parameterized by such a homeomorphism. To make things unique, let us assume that the parameter space S^1 is traversed clockwise and that the homeomorphism is orientation preserving. A simple closed curve is called a *convex closed curve* if it encloses a convex set.

On the other hand, the boundary ∂C of any bounded convex set $C \subset \mathbb{R}^2$

is a closed convex curve. The following lemma states some facts about the Hausdorff distance of convex sets in arbitrary dimension, where for $A \subseteq \mathbb{R}^d$ $ch(A)$ denotes the *convex hull* of A.

Lemma 1. *Let $A, B \subset \mathbb{R}^d$ be bounded sets. Then*

$$\delta_H(\partial ch(A), \partial ch(B)) = \delta_H(ch(A), ch(B)) \leq \delta_H(A, B).$$

Proof. To show the equality let $\chi = \delta_H(ch(P), ch(Q))$, and let $a \in ch(A)$ and $b \in ch(B)$ with $||a - b|| = \chi$. Since the intersection of the ball around b with radius χ with $ch(A)$ contains only the point a, a must lie on the boundary of $ch(A)$. Likewise, b must lie on the boundary of $ch(B)$. This shows that $\delta_H(\partial ch(A), \partial ch(B)) \geq \delta_H(ch(A), ch(B))$. On the other hand suppose that $a \in \partial ch(A)$ and that b is the point of $ch(B)$ closest to a, therefore $||a - b|| \leq \chi$. As before, we can argue that b must lie on the boundary of $ch(B)$, which shows that $\delta_H(\partial ch(A), \partial ch(B)) \leq \chi$.

To prove $\delta_H(ch(A), ch(B)) \leq \delta_H(A, B)$ let $\delta = \delta_H(A, B)$ and $a \in ch(A)$. Then a can be represented as a convex combination $a = \sum_{i=0}^k c_i a_i$ with $a_1, ..., a_k \in A$. Let $b_1, ..., b_k \in B$ such that $||a_i - b_i|| \leq \delta$ for $i = 1, ..., k$. Then $a = \sum_{i=0}^k c_i a_i = \sum_{i=0}^k c_i(b_i + e_i) = \sum_{i=0}^k c_i b_i + \sum_{i=0}^k c_i e_i = b + e$, where $e_i = a_i - b_i$ for $i = 1, ..., k$. $b \in ch(B)$, since it is a convex combination of elements of B. $||a - b|| = ||e|| \leq \delta$ since e is a convex combination of vectors of length $\leq \delta$. $\square$

We define the *Fréchet distance* of two simple closed curves as in Definition 1 only that ρ and σ are orientation preserving homeomorphisms on S^1.

Closed curves have the following property:

Lemma 2. *Let P and Q be simple closed curves, φ a homeomorphism from P to Q, and $\delta \geq 0$ such that $||a - \varphi(a)|| \leq \delta$ for all a on P. Then $\delta_F(P, Q) \leq \delta$.*

To see this, consider Definition 1 and let ρ be the identity on S^1 and $\sigma = Q^{-1} \circ \varphi \circ P$. Then for all $t \in S^1$ $||P(\rho(t)) - Q(\sigma(t))|| = ||P(t) - \varphi(P(t))|| \leq \delta$ which implies that $\delta_F(P, Q) \leq \delta$.

The main result of this section is

Theorem 1. *For any pair of closed convex curves P and Q,*

$$\delta_H(P, Q) = \delta_F(P, Q).$$

In order to prove the main result, we first show the following lemma.

Lemma 3. *Let P, Q be closed convex curves and R the boundary of $ch(P \cup Q)$. Then to any point c on R that does not lie in $P \cup Q$ there exist points a on P and b on Q such that c lies on the line segment $\overline{ab}$.*

Proof. Since c lies in the convex hull of $P \cup Q$ it can be represented as a convex combination $c = \sum_{i=0}^{n} a_i x_i$ with $x_i \in P \cup Q$ for $i = 1, ..., n$. Separating the $x_i \in P$ from the ones in Q we obtain a representation $c = \lambda a + (1 - \lambda) b$ for some $a \in ch(P)$, some $b \in ch(Q)$, and $\lambda \in [0, 1]$, i.e., c lies on the line segment $\overline{ab}$. Neither a nor b can lie in the interior of $ch(P)$ or $ch(Q)$ since that would imply that c lies in the interior of R, so $a \in P$ and $b \in Q$. □

Now let us get to the proof of Theorem 1

Proof. We will assume in the proof that P and Q are smooth curves. The result follows for arbitrary convex curves then by reasons of continuity, since they can be approximated arbitrary closely by smooth curves. Let R be the convex, closed curve which forms the boundary of $ch(P \cup Q)$.

Let n_P be the function mapping any point a on P to the intersection point of the ray normal to P in a with R (see Figure 3) . Observe, that

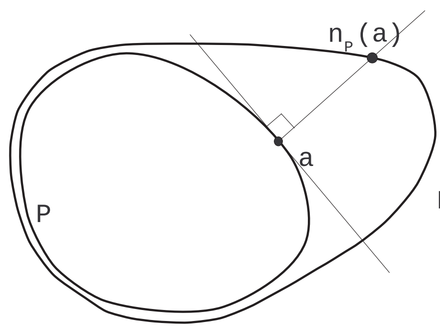

Figure 3: The function n_P.

a is the closest point of the point $n_P(a)$ on P. a is unique because of the smoothness of P. Therefore, n_P is bijective and, in fact, a homeomorphism from P to R. Analogously, we define the function n_Q. Then $\varphi = n_Q \circ n_P^{-1}$ is a homeomorphism from P to Q. We will prove the following claim for $\delta = \delta_H(P, Q)$

Claim $||\varphi(a) - a|| \leq \delta$ for all a on P.

To see this, let a be an arbitrary point on P, $b = \varphi(a)$, and $c = n_P^{-1}(a)$, so $c = n_Q(b)$. If c lies on P then $c = a$ and b is the point closest to a on Q. If c lies on Q then $c = b$ and a is the point closest to b on P. In both cases the claim follows from the definition of the Hausdorff distance.

It remains to look at the case where c does not lie on P or Q, see Figure 4. Let t_P and t_Q be the tangents to P and Q in a and b, respectively. By Lemma 3 there exist points a'' on P and b'' on Q such that c lies on the line segment $\overline{a''b''}$. Let a' and b' be the intersection points of t_P and t_Q with $\overline{a''b''}$, then c lies on the line segment $\overline{a'b'}$, as well. To each point x on this line

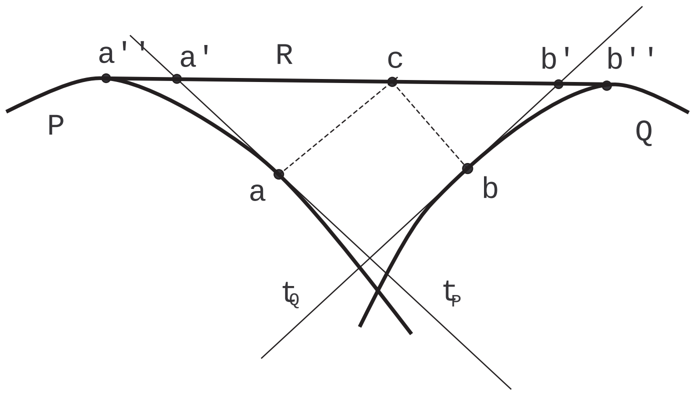

Figure 4: Homeomorphism from P to Q.

segment we assign a value $f(x) = d(x, t_P) + d(x, t_Q)$, the sum of distances of x to t_P and t_Q. Observe, that $f(a') = d(a', t_Q) \leq \delta$. This follows from the definition of Hausdorff distance since any point on Q has distance at least $d(a', t_Q)$ from a'. Likewise, $f(b') \leq \delta$. Consequently, since f is an affine function its value is bounded by δ for any point on the line segment $\overline{a'b'}$, in particular, $f(c) \leq \delta$. Since $||a - b|| \leq f(c)$ by the triangle inequality, the claim follows.

The claim, in combination with Lemma 2, implies that $\delta_F(P, Q) \leq \delta$ which proves the theorem. □

We obtain the following corollary to Theorem 1 which is actually a generalization and states that for two curves that are "nearly convex" the Hausdorff and Fréchet distance are close, as well.

Corollary 1. *Let P and Q be two simple closed curves and $\partial ch(P)$ and $\partial ch(Q)$ the boundaries of their convex hulls. Then $\partial ch(P)$ and $\partial ch(Q)$ are convex closed curves and the following inequation holds*

$$\delta_F(P, Q) \leq \delta_H(P, Q) + \delta_F(P, \partial ch(P)) + \delta_F(Q, \partial ch(Q))$$

To see this observe, that by the triangle inequality $\delta_F(P, Q) \leq \delta_F(P, \partial ch(P)) + \delta_F(\partial ch(P), \partial ch(Q)) + \delta_F(Q, \partial ch(Q))$. Furthermore, $\delta_F(\partial ch(P), \partial ch(Q)) = \delta_H(\partial ch(P), \partial ch(Q)) \leq \delta_H(P, Q)$ by Theorem 1 and Lemma 1. Both inequalities together prove the corollary.

3 κ-straight curves

Let us now turn our attention to *κ-straight* curves; for these curves the arclength between any two points is at most a constant κ times their Euclidean distance.

Definition 2 (**κ-Straightness**). *A rectifiable planar curve P is called κ-* straight *for some real parameter $\kappa \geq 1$, if the following holds for any two points x and y on P:*

$$\mathrm{arc}_P(x, y) \leq \kappa \cdot ||x - y||,$$

where $\mathrm{arc}_P(x, y)$ *is the arclength of the piece of P between x and y.*

Examples for straight curves are the curves with increasing chords of [12], where $\kappa \leq 2\pi/3$, or the self-approaching curves of [1]. The aim of this paper is to show that the Fréchet distance of κ-straight curves differs from the Hausdorff distance by at most a factor of $\kappa + 1$, if the endpoints are close enough. A preliminary version of this result was presented in [5]. Rather than for κ-straight curves we will here prove the result for so-called *κ-bounded curves*, which is a generalization of κ-straight curves. We will investigate the connection of both in the next section.

For $p \in \mathbb{R}^2$ and $r > 0$ let $D(p, r)$ be the closed disk centered at p with radius r.

Definition 3 (**κ-Boundedness**). *A planar curve P is called* κ-bounded *for some real parameter $\kappa \geq 1$, if the following holds for any two points $x = P(s)$ and $y = P(t)$ with $s < t$ on P:*

$$P|_{[s,t]} \subseteq D(x, \frac{\kappa}{2}||x - y||) \cup D(y, \frac{\kappa}{2}||x - y||)\,.$$

See Figure 5 for an illustration.

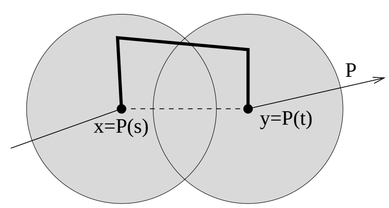

Figure 5: Example of a κ-bounded curve for $\kappa = 3/2$.

Lemma 4. *Every κ-straight curve is also κ-bounded. For every $\kappa > 1$ there are κ-bounded curves which are not κ-straight.*

Proof. Let f be a κ-straight curve, and let x, y be two points on f. If the piece of f between x and y would leave $D(x, \frac{\kappa}{2}||x - y||) \cup D(y, \frac{\kappa}{2}||x - y||)$, then $\mathrm{arc}_f(x, y) > \kappa \cdot ||x - y||$, which is a contradiction to the κ-straightness of f. Thus f is κ-bounded.

For the second claim we construct a κ-bounded curve f which is not κ-straight, c.f. Figure 6. Fix two arbitrary points $x, y \in \mathbb{R}^2$, $x \neq y$, and let

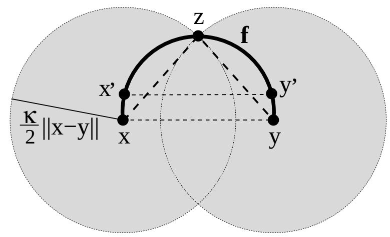

Figure 6: A κ-bounded curve which is not κ-straight.

$z \in \mathbb{R}^2$ be one of the two points such that $||z-x|| = ||z-y|| = \kappa/2||x-y||$. Consider the uniquely defined circle through x, y, z. We define f to be the arc of this circle that starts in x, then passes z, and ends in y. $\mathrm{arc}_f(x,y)$ is larger than the arc length of the dashed curve in Figure 6, which is $\kappa||x-y||$. Thus f is not κ-straight. Let $\overline{xy}$ be the line segment between x and y. Now we need to check the boundedness condition for all x', y' on f. Note that since f is a circular arc it suffices to check only all x', y' on f with $\overline{x'y'}$ parallel to $\overline{xy}$. The κ-boundedness now follows from similarity of triangles: Consider the bisector to $\overline{x'y'}$, and let $d(x',y')$ be the distance between z and the intersection of this bisector with $\overline{x'y'}$, see Figure 7.

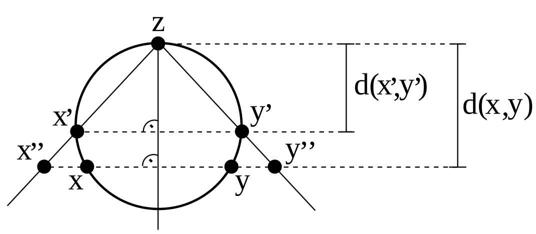

Figure 7: The boundedness condition for x', y'.

It suffices to show that $2d(x',y') \leq ||x'-y'||\sqrt{\kappa^2-1}$, because this implies that $||z-x'|| = ||z-y'|| \leq \kappa/2||x'-y'||$. Inspecting Figure 7 we see that

$$\frac{d(x',y')}{||x'-y'||} = \frac{d(x,y)}{||x''-y''||} \leq \frac{d(x,y)}{||x-y||} = \frac{1}{2}\sqrt{\kappa^2-1}$$

□

We will see below that the boundedness property also rules out the possibility of curves with small Hausdorff distance that have a large Fréchet distance: In Section 4 we show that the Fréchet distance of κ-bounded curves is at most $(\kappa+1)$ times their Hausdorff distance. This result gives rise to a randomized approximation algorithm that computes an upper bound on the Fréchet distance between two κ-bounded curves that is off from the exact value by a multiplicative factor of $(\kappa + 1)$. The algorithm is given in Section 5 and runs in $\mathcal{O}((m+n)\log^2(m+n)2^{\alpha(m+n)})$ time for given polygonal curves P, Q with n and m vertices, and thus outperforms the fastest known algorithm to compute the Fréchet distance exactly, which requires $\mathcal{O}(mn \log(mn))$ time, see [4]. Here $\alpha(n)$ is the functional inverse of the Ackermann function, see [13].

4 The upper bound

In this section we will show that for κ-bounded polygonal curves the Fréchet distance is at most a factor of $(\kappa+1)$ away from the Hausdorff distance.

In the following, $P\colon [0,n] \to \mathbb{R}^2$ and $Q\colon [0,m] \to \mathbb{R}^2$ will be *polygonal curves* with $n+1$ and $m+1$ vertices, respectively, such that for all $0 \le i < n$ and for all $0 \le j < m$ each $P|_{[i,i+1]}$ and each $Q|_{[j,j+1]}$ is affine.

Definition 4 (Alt/Godau, [4]). *The set* $F_\delta(P,Q) := \{(x,y) \in [0,n] \times [0,m] \mid ||P(x) - Q(y)|| \le \delta\}$*, or* F_δ *for short, is called the* free space *of* P *and* Q*.*

Sometimes we refer to $[0,n]\times[0,m]$ as the *free space diagram*; the *feasible* points $p \in F_\delta$ will be called ‘white’ and the *infeasible* points $p \in [0,n]\times[0,m] \setminus F_\delta$ will be called ‘black’ (for obvious reasons, c.f. Figure 9). By L and R,

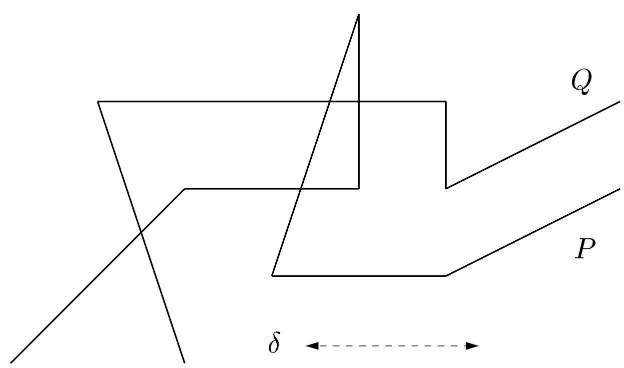

Figure 8: Two polygonal curves P and Q.

we will denote the lower left and the upper right corner of F_δ, respectively, i.e., $L := (0,0)$, and $R := (n,m)$.

As is shown in [4], for polygonal curves P and Q we have that $\tilde{\delta}_F(P,Q) \le \delta$ iff there exists a path within F_δ from L to R. We have that $\delta_F(P,Q) \le \delta$ iff there exists a path within F_δ from L to R which is monotone in both coordinates (such a path will be called *bi-monotone*). And we have that $\delta_H(P,Q) \le \delta$ iff there exists for each $x \in [0,n]$ a $y \in [0,m]$ such that $(x,y) \in F_\delta$ is white and vice versa.

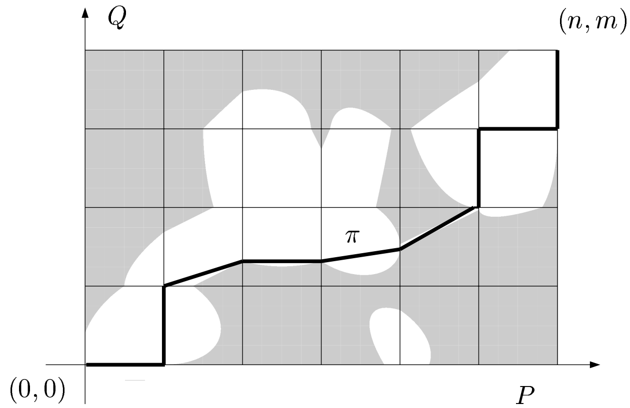

Figure 9: The free space diagram of the curves from Figure 8 for a given δ. An example path π in the free space is drawn bold.

Theorem 2. *For any pair of κ-bounded polygonal curves P, Q*

$$\delta_F(P, Q) \leq (\kappa + 1)\delta_H(P, Q),$$

if $\max(||P(0) - Q(0)||, ||P(n) - Q(m)||) \leq \delta_H(P, Q)$.

Proof. Set $\delta = \delta_H(P, Q)$. Since both $||P(0) - Q(0)||$ and $||P(n) - Q(m)||$ are bounded by δ, we know that the points L and R in F_δ are white.

To each $x \in [0, n]$ we assign the value $h(x)$ defined by

$$h(x) = \max\{y \mid \exists x' \leq x \text{ s.t. } (x', y) \in F_\delta\}.$$

h is a piecewise continuous monotone function consisting of of elliptic arcs and horizontal segments. At the points of discontinuity we add vertical segments and obtain a curve φ_δ from the graph of h (see Figure 10). In addition, we add the vertical segment $\overline{Lh(0)}$ such that φ_δ starts in L and ends in R.

By definition of φ_δ there is no white point in F_δ in the region above φ_δ.

Since φ_δ is a bi-monotone path from L to R in F_δ we can parametrize it as $\varphi_\delta(t) = (\rho(t), \sigma(t)), t \in [0, 1]$ with increasing functions $\rho : [0, 1] \to [0, n]$ and $\sigma : [0, 1] \to [0, m]$. We can use ρ and σ as reparametrizations of P and Q. We claim that these reparametrizations yield a Fréchet distance of at most $(\kappa+1)\delta$. For this we have to show that $||P(\rho(t)) - Q(\sigma(t))|| \leq (\kappa+1)\delta$ for each $t \in [0, 1]$.

If $(\rho(t), \sigma(t))$ is white we know that $||P(\rho(t)) - Q(\sigma(t))|| \leq \delta$. By definition of φ_δ, black points on φ_δ are always part of black horizontal or vertical open line segments on φ_δ. Furthermore, the left endpoint of a black horizontal line segment is always white, and so is the upper endpoint of a vertical black line segment. Now consider a black point (x_1, y_0) on φ_δ that lies

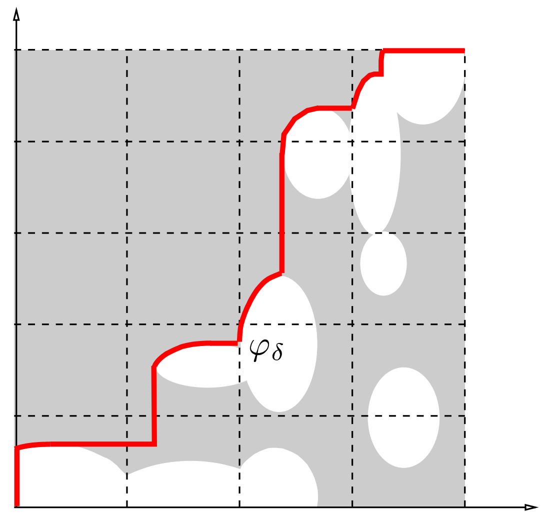

Figure 10: The reparametrization φ_δ in F_δ.

w.l.o.g. on a horizontal black segment with the white left endpoint (x_0, y_0), c.f. Figure 11.

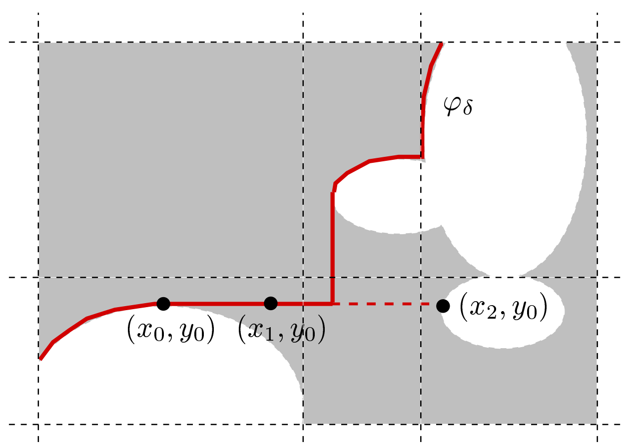

Figure 11: A horizontal black segment with the white left endpoint (x_0, y_0) in F_δ.

It remains to show that $||P(x_1) - Q(y_0)|| \leq (\kappa + 1)\delta$.

Since $\delta \geq \delta_H(P, Q)$ there exists for each y an x such that (x, y) is white. From the definition of φ_δ we know that the region above φ_δ is black, so $x > x_0$ if $y > y_0$. But since the white regions in F_δ are closed by definition, we can conclude that there exists another white point (x_2, y_0) on the horizontal ray emanating from (x_0, y_0) and crossing (x_1, y_0).

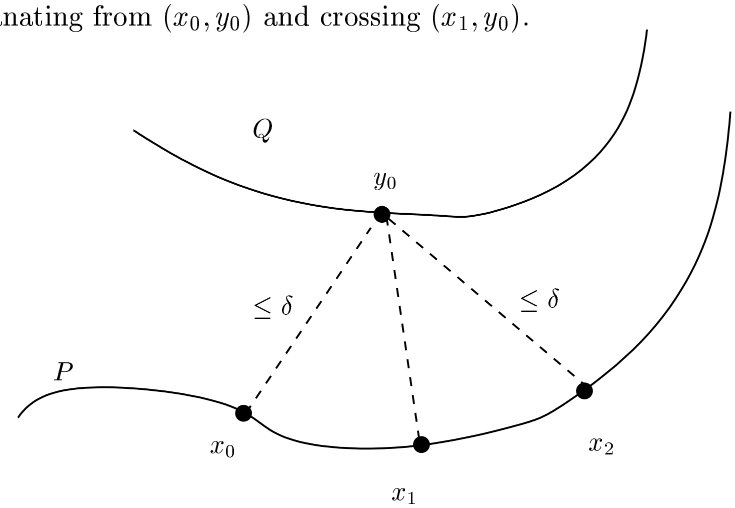

Figure 12: The corresponding geometric situation.

The geometric situation corresponding to this constellation is depicted in Figure 12. By applying the triangle inequality twice we obtain

$$\begin{aligned} & ||P(x_1) - Q(y_0)|| \\ \leq\ & \min(||P(x_0) - Q(y_0)|| + ||P(x_0) - P(x_1)||, \\ & \quad ||P(x_2) - Q(y_0)|| + ||P(x_1) - P(x_2)||) \\ \leq\ & \delta + \min(||P(x_0) - P(x_1)||, ||P(x_1) - P(x_2)||) \ . \end{aligned}$$

From the κ-boundedness property it follows that the min-term is bounded from above by $\frac{\kappa}{2}||P(x_0) - P(x_2)||$. Applying again the triangle inequality $||P(x_0) - P(x_2)|| \leq ||P(x_0) - Q(y_0)|| + ||Q(y_0) - P(x_2)||$ we obtain $||P(x_1) - Q(y_0)|| \leq \delta + \frac{\kappa}{2}(\delta + \delta) = (\kappa + 1)\delta$. □

With the same proof we obtain a result for the weak Fréchet distance: When we set $\delta := \tilde{\delta}_F(P, Q)$ in the proof of Theorem 2, we know that $\max(||P(0) - Q(0)||, ||P(n) - Q(m)||) \leq \tilde{\delta}_F(P, Q)$, and $\delta \geq \delta_H(P, Q)$. Thus we obtain:

Corollary 2. *For any pair of κ-bounded polygonal curves P, Q*

$$\delta_F(P, Q) \leq (\kappa + 1)\tilde{\delta}_F(P, Q).$$

5 Computing the reparametrization

In this section we describe a divide-and-conquer algorithm that computes the reparametrization φ_δ. We show the following result.

Theorem 3. *Let P, Q be* simple *polygonal curves with $n + 1$, and $m + 1$ vertices, and $\delta > 0$. Then φ_δ can be computed in $\mathcal{O}((m + n)\log^2(m +$*

$n)2^{\alpha(m+n)})$ *time by a randomized algorithm, and in* $\mathcal{O}((m+n)\log^3(m+n)2^{\alpha(m+n)})$ *time by a deterministic one.*

Proof. The algorithm proceeds as follows: In a first step, we determine $Q(y_{1/2})$ the point on Q that corresponds to the intersection of φ_δ with the vertical line $x = n/2$ in F_δ that corresponds to the midpoint $P(n/2)$ of P. Then we split Q into $Q_B := Q|_{[0,y_{1/2}]}$ and $Q_T := Q|_{[y_{1/2},m]}$ and P into $P_L := P|_{[0,n/2]}$ and $P_R := P|_{[n/2,n]}$. We recursively compute $\varphi_\delta(P_L, Q_B)$ and $\varphi_\delta(P_R, Q_T)$, and glue them together, c.f. Figure 13.

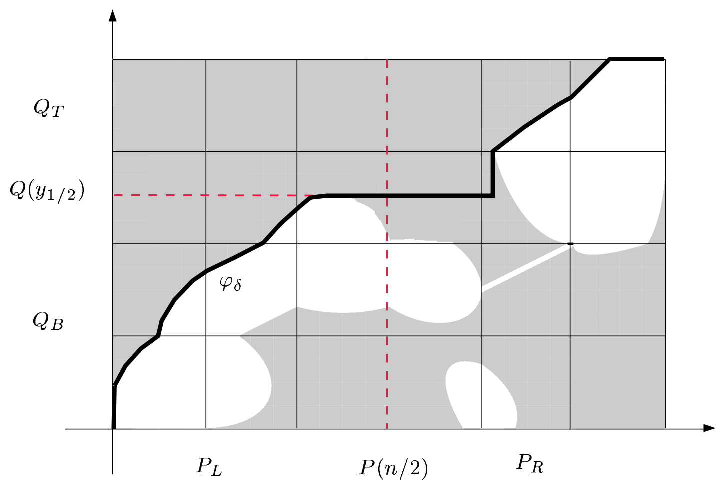

Figure 13: The recursion step.

We will argue below that we can compute $Q(y_{1/2})$ in $\mathcal{O}((m+n)\log^2(m+n)2^{\alpha(m+n)})$ deterministic, and $\mathcal{O}((m+n)\log(m+n)2^{\alpha(m+n)})$ randomized time. This proves the claimed time bounds since the total size of all the subproblems is $\mathcal{O}(m+n)$ on each level of the recursion, and the recursion terminates in $\mathcal{O}(\log n)$ steps.

In order to compute $Q(y_{1/2})$, we imagine the following process: First we cut off the right half of F_δ, i.e., we only look at $F_\delta(P_L, Q)$. Now we start sweeping a horizontal line from $y = m$ downwards to $y = 0$ until it hits the first white point. This will happen at $y_{1/2}$. Of course we cannot afford to compute the diagram F_δ (or larger parts of it) explicitly, so we have to proceed in a different way.

In a first step we check in $\mathcal{O}(n)$ time, if the endpoint $Q(m)$ is δ-close to P_L. If this is the case, we are already finished. Otherwise we consider $\mathsf{nh}_\delta(P_L)$, the δ-neighborhood of P_L. This is the set of all points that have distance at most δ to P_L. This set can be described as a union of rectangles of width δ and circles of radius δ. The boundary $\mathsf{bd}_\delta(P_L)$ of $\mathsf{nh}_\delta(P_L)$ consists of circular arcs (of radius δ) and line segments. Since P is simple, this boundary has complexity $\mathcal{O}(n)$ (c.f. [10] and [11]), and can be computed in

$\mathcal{O}(n \log n)$ time, (c.f. [6]). The basic idea of our algorithm is based on the following simple observation:

The first point $Q(y_{1/2})$ on Q (starting from $Q(m)$) that intersects $\mathsf{bd}_\delta(P_L)$ is part of the boundary of the cell C of the arrangement $\mathbf{A}$ induced by Q and $\mathsf{bd}_\delta(P_L)$, that contains $Q(m)$, c.f. Figure 14.

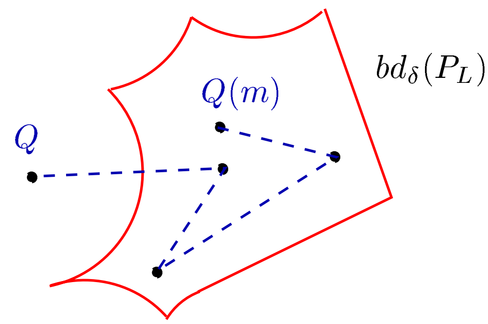

Figure 14: The cell C in $\mathbf{A}$.

If we consider the segments and circular arcs of $\mathsf{bd}_\delta(P_L)$ as a set of $\mathcal{O}(n)$ *red*, and the segments of Q as a set of $\mathcal{O}(m)$ *blue* Jordan arcs, we can conclude with the Combination Lemma of [9] (see also [13], Lemma 6.7), that the cell C has complexity $\mathcal{O}(m+n)$.

So all we have to do is to compute the cell C, and check its boundary. For technical reasons, we need a point $\mathbf{q}$ in this cell, that is neither part of Q nor of $\mathsf{bd}_\delta(P_L)$ (in particular we cannot use $Q(m)$ itself). Since any two of the arcs intersect at most twice, the cell of $\mathbf{A}$ that contains the point $\mathbf{q}$ can be computed in $\mathcal{O}(\lambda_4(m+n)\log^2(m+n)) = \mathcal{O}((m+n)\log^2(m+n)2^{\alpha(m+n)})$ deterministic time ([9], see also [13], Theorem 6.11), or in $\mathcal{O}(\lambda_4(m+n)\log(m+n)) = \mathcal{O}((m+n)\log(m+n)2^{\alpha(m+n)})$ randomized time ([7], see also [13], Theorem 6.15), where $\lambda_4(n) = \mathcal{O}(n2^{\alpha(n)})$ is the maximum length of a Davenport-Schinzel sequence of order 4 over an n-element alphabet. □

With an algorithm of Alt et al. [2] we can compute $\delta = \delta_H(P,Q)$ in $\mathcal{O}((m+n)\log(m+n))$ time. Combining this with Theorems 2 and 3, we can find a $(\kappa+1)$-approximation to $\delta_F(P,Q)$, together with a reparametrization φ_{app} that witnesses this fact, within the time bounds as stated in Theorem 3.

Corollary 3. *For any pair of κ-bounded polygonal curves P, Q with $n+1$ and $m+1$ vertices, respectively and $\max(||P(0)-Q(0)||, ||P(n)-Q(m)||) \leq \delta_H(P,Q)$, we can compute a $(\kappa+1)$-approximation to $\delta_F(P,Q)$, together with a reparametrization φ_{app} in $\mathcal{O}((m+n)\log^2(m+n)2^{\alpha(m+n)})$ time by a randomized algorithm, and in $\mathcal{O}((m+n)\log^3(m+n)2^{\alpha(m+n)})$ time by a deterministic one.*

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