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# Optimal PNNI complex node representations for restrictive costs

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**Abstract**

The Private Network-to-Network Interface (PNNI) is a scalable hierarchical protocol that allows ATM switches to be aggregated into clusters called peer groups. To provide good accuracy in choosing optimal paths in a PNNI network, the PNNI standard provides a way to represent a peer group with a structure called the complex node representation. It allows the cost of traversing the peer group between any ingress and egress to be advertised in a compact form. Complex node representations using a small number of links result in a correspondingly short path computation time and therefore in good performance. It is, accordingly, desirable that the complex node representation contains as few links as possible. In earlier work, a method was presented for constructing the set of the optimal complex node representations in the restrictive and symmetric cost case, under the assumption of a restricted set of optimal paths and a corresponding minimal path computation time. Here this method is extended to constructing the set of the optimal complex node representations appropriate for deployment in a heterogeneous environment where no uniform policy is used to derive them. These representations are not confined by a reduced optimal path constraint, and consequently use the absolute minimum possible number of links, resulting in a minimum path computation time.



*Keywords:* Complex node representation; Private Network-to-Network Interface; Restrictive cost; State aggregation; Topology aggregation

## 1. Introduction

Private Network-to-Network Interface (PNNI) is a hierarchical, dynamic link-state routing protocol defined by the ATM Forum for use between private ATM switches as well as between groups of private ATM switches [1]. It is a scalable protocol that clusters network nodes into manageable groups called *peer groups*. The details of a peer group are abstracted into one logical node. This method can be applied recursively so that PNNI can hierarchically aggregate network-topology state information. The hierarchical aggregation of network-topology state information results in a reduction of the overall complexity, and, in particular, in a reduction of the amount of memory and time required to compute paths through the network. The PNNI routing hierarchy is designed to reduce this overhead while providing efficient routing.

To provide good accuracy in choosing optimal paths in a PNNI network, the PNNI standard provides a way to represent a peer group with a structure called the *complex node representation*. The complex node representation allows the cost of traversing the corresponding logical node to be advertised, which represents the cost of traversing the summarized peer group. An alternative representation using a spanning tree structure was presented in Refs. [2,3]. A method to generate a complex node representation corresponding to a given peer group has been specified in Ref. [4] for the case of symmetric and restrictive costs, such as available bandwidth. Both this and the spanning tree representation are exact representations in that they capture the full details of the underlying peer group topology. However, representations that are not necessarily exact can be obtained based on the methods presented in Refs. [5–9]. These works consider symmetric and restrictive as well as additive costs, such as delay, whereas the case of asymmetric costs is treated in Ref. [10]. An approach for determining the parameters of complex node representations satisfying various optimal objective functions is presented in Ref. [11].

In order to facilitate the path selection algorithm and minimize the path computation time, it is desirable that the complex node representation contains as few links as possible. Furthermore, for efficiency reasons and in order to

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reduce the topology update flooding in PNNI, complex node representations are not computed every time a cost change occurs, but only at the instants when significant changes take place. The issue of how to generate optimal complex node representations at these instants, in the case of symmetric and restrictive costs, and under the assumption of minimal path computation time was addressed in Ref. [4]. Each state parameter associated with a link was assumed to be the same in both directions of the link. Restrictive costs correspond to the case where the measure of interest is, for example, the available bandwidth. The complex node representations obtained were exact and optimal (in the sense that they used the minimum number of links) within the class of complex node representations for which the path computation time is minimal. More specifically, the complex node representations belonging to this class had the property that a restricted set of two paths in the representation (namely, either a one-hop path using an exception bypass or a two-hop path obtained by the concatenation of two spokes) always contained an optimal path. Because an optimal path can be identified by considering only these two paths, we shall refer to this class of representations as *restricted optimal path complex node representations*. Note that these representations were developed for deployment in the context of a homogeneous environment, where all representations belong to the same class and all have the same properties.

Unlike in a homogeneous environment, in a heterogeneous environment every peer group is allowed to use its own policy to determine the corresponding complex node representation. In such an environment, therefore, there are no means of knowing whether complex node representations belong to the above-mentioned class or not. By assuming that they all belong to this class, the final outcome may be a path that is locally optimal at some peer groups but globally sub-optimal owing to the lack of a uniform policy in deriving complex node representations. More specifically, for an arbitrary complex node representation, one of the two paths described above may not necessarily be an optimal one. Consequently, all possible paths should be considered in order to identify the optimal one, which means that the minimal path computation time property of the restricted optimal path complex node representations can no longer be exploited. As a result, when these representations are used in a heterogeneous environment, the benefit of minimizing the path computation time vanishes. As all paths should be considered, again to facilitate the path selection algorithm and minimize the path computation time, it is desirable that the complex node representation obtained contains as few links as possible, albeit without the restriction on optimal paths imposed in Ref. [4]. Removing this restriction, on the one hand, increases the set of paths and, consequently, the path computation time compared with the case of a homogeneous environment, whereas on the other hand it potentially decreases the number of links required by an exact optimal complex node representation as the set of complex node representations increases. Therefore, in the context of a heterogeneous environment, this will result in a decreased path computation time compared with the time required when the restricted optimal path complex node representations are deployed.

This paper presents a method for constructing the set of the optimal complex node representations, which are not confined by a reduced path constraint, and consequently use the absolute minimum possible number of links. Similarly to the method developed in Ref. [4], the establishment of the optimal substructure property of the optimal complex node representations is central to the development of this method. This implies that the optimal solution to the original problem is derived from the optimal solutions of appropriately identified sub-problems. In Section 2, the basic definitions of node representations in the context of PNNI are given, and the notions related to the cost transition matrix are briefly reviewed. In Section 3, the notion of the group evolution process is reviewed, and the concept of the spanning line representation is introduced. The basic definitions associated with the complex node representations are given in Section 4, and the method for constructing the set of the optimal complex representations is derived in Section 5. Section 6 presents a numerical example, whereas Section 7 contains the derivation of the bounds on the number of links of the resulting set of optimal complex node representations.

## 2. PNNI node representations

A key feature of the PNNI protocol is the ability to cluster network nodes into manageable groups called peer groups. This concept is illustrated schematically in Fig. 1. The PNNI peer group shown is composed of six nodes. Nodes 1–4 are called *border nodes* because they connect the peer group to other peer groups.

ATM is a source routing technology. To enable source route computation and to support end-to-end QoS (for example the required bandwidth), the nodes must maintain information about the network-topology. PNNI thus defines a system for the creation and distribution of topology data within a network so that each node can maintain a topology database, which defines its view of the network. This allows nodes to select paths for routing calls through the network, and to perform alternative routing in the case of link failure.

The topology data required for path selection and routing may include not only details of the layout of nodes and links but also QoS parameters as mentioned above. For example, a call to be routed over the network may require a certain bandwidth. In this case, knowledge of the available bandwidth of links in the network is required to determine if a call can be established successfully. To allow such parameters to be taken into account, *costs* can be associated with links and paths in

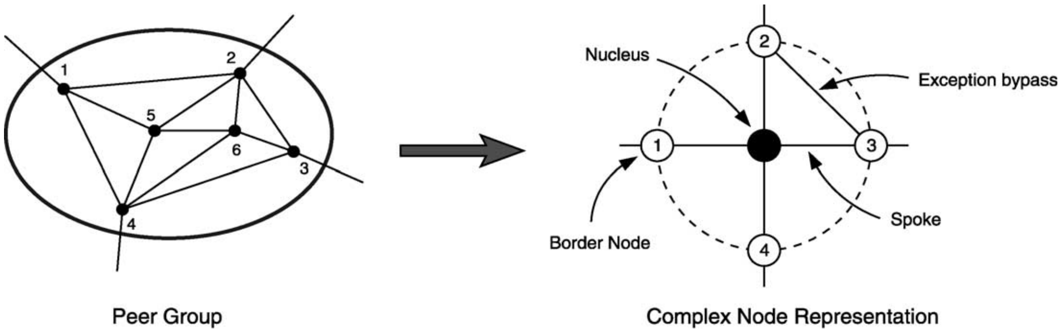



Fig. 1. Peer group and complex node representation.

the network. The cost of a link is expressed as an arbitrary value determined as some function of the parameter, e.g. available bandwidth about which knowledge is required. Whatever be the particular function employed, according to convention it is usual for the cost to be defined such that the lower the cost the better the link. In the case of bandwidth, for example, the cost $C$ of a link may be defined as the inverse of the (available) bandwidth (i.e. $C = 1/\text{bandwidth}$), or as the difference $C = (\text{constant}) - \text{bandwidth}$, with the constant being equal to the maximum bandwidth of all the links [4].

A path in the network, involving multiple links, can be measured by a *restrictive cost*. According to the definition of restrictive cost, the weakest link in a path defines the restrictive cost of the path. Thus, when convention is followed such that a higher cost corresponds to a weaker link, the restrictive cost of a path will be determined by the maximum of the costs of the constituent links.

To allow such costs to be taken into account in the path selection process, PNNI provides a way to represent a peer group as a logical group node called *complex node representation*. As discussed further below, a peer group can be modeled by an orientated graph in which a node of the peer group is referenced as a vertex of the graph, and a link between nodes is referenced as an edge between two vertices of the graph. As shown in Fig. 1, a complex node representation consists of a number of vertices corresponding to the border nodes of the peer group, as well as a nucleus vertex. The nucleus is connected to the border vertices through *spokes*, and optionally, border vertices can be directly connected by *exception bypasses*.

The complex node representation is derived using a set of restrictive costs for the peer group which is usually presented in the form of a cost matrix known as the *transition matrix* for the peer group. The transition matrix defines the restrictive costs of optimal (lowest-cost) paths between all pairs of border nodes in the peer group. Let $M_N(C)$ be the cost transition matrix corresponding to a peer group containing $N$ border vertices $b_1, \ldots, b_N$,

$$M_N(C) = \begin{bmatrix} 0 & c_{1,2} & \cdots & c_{1,N} \\ c_{2,1} & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ & & & c_{N-1,N} \\ c_{N,1} & \cdots & c_{N,N-1} & 0 \end{bmatrix}$$

where $c_{i,j}$ denotes the cost of the optimal path between the border vertices $b_i$ and $b_j$. Owing to the cost symmetry, this matrix is symmetric.

The complex node representation, derived on the basis of the transition matrix, indicates the cost of traversing the peer group, and therefore allows such costs to be taken into account for path selection purposes. In particular, there may be many possible complex node representations corresponding to a given transition matrix and hence a given peer group. In order to minimize the path computation time, which is closely related to the connectivity of the complex node representation, it is desirable that the complex node representation is optimized as far as possible by minimizing the number of its edges. Note that, because the number of spokes is fixed and equal to the number of border nodes, minimizing the number of edges is equivalent to minimizing the number of exception bypasses. However, it is also important to ensure that the resulting representation accurately reflects the transition matrix.

## 3. Matrix properties

In this section we briefly review the notion of the *group evolution* process introduced in Ref. [4]. This process is of significant interest because as shown in Section 5, it is closely coupled with the structure of the optimal complex node representations. A detailed numerical example illustrating the group evolution process is presented in Section 6. Furthermore, we briefly review the spanning tree representation [3] and then introduce the *spanning line* representation obtained from the group evolution process. As shown in Section 5, these representations are used to derive the optimal complex node representations.

### 3.1. Group evolution process

Let $c_{\min}(c_{\max})$ be the minimum (maximum) transition cost corresponding to the transition matrix $M_N(C)$. Formally,

$$c_{\min} = \min_{\substack{\forall(i,j)\\ i\neq j}} \{c_{i,j}\}, \qquad c_{\max} = \max_{\substack{\forall(i,j)\\ i\neq j}} \{c_{i,j}\}. \tag{1}$$

Also let $F$ be the number of different entry costs contained in matrix $M_N(C)$ in increasing order:

$$c_{\min} = C_1 < C_2 < \cdots < C_k < \cdots < C_F = c_{\max}. \tag{2}$$

According to the group evolution process, the values of the restrictive costs in the transition matrix are considered in the order specified above. At the $k$th iteration, the set $G_k$ of groups $G_k^{(1)},\ldots,G_k^{(g_k)}$ corresponding to the cost $C_k$ is obtained. A typical such group $G_k^{(m)}$ $(1 \le m \le g_k)$ is characterized by the following properties:

$$\forall(n_i,n_j) : n_i \in G_k^{(m)},\ n_j \in G_k^{(m)} \text{ it holds that } c_{i,j} \le C_k, \tag{3}$$

and

$$\forall(n_i,n_j,n_p) : n_i \in G_k^{(m)},\ n_j \in G_k^{(m)},\ n_p \notin G_k^{(m)}$$
$$\text{it holds that } c_{i,p} = c_{j,p} > C_k. \tag{4}$$

The first, second and last iteration of the group evolution process are schematically depicted in Fig. 2.

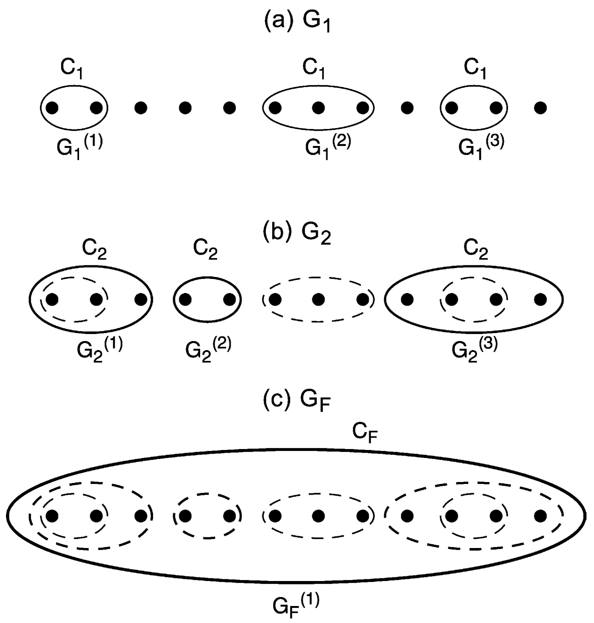



Fig. 2. Group evolution.

### 3.2. Spanning tree and spanning line representations

For a network consisting of $N$ border nodes, any representation connecting all of the $N$ nodes uses at least $N-1$ links. An accurate representation can be obtained using a spanning tree consisting of $N-1$ links [3]. Obviously, this representation is optimal because it uses the minimum possible number of links. In this section we demonstrate that the group evolution process always allows us to obtain an optimal representation using a spanning line. Also note that a spanning line is a special degenerate case of a spanning tree.

First, we consider the $G_1$ set consisting of the groups $G_1^{(1)},\ldots,G_1^{(g_1)}$. The nodes contained in a typical such group $G_1^{(m)}(1 \le m \le g_1)$ are ordered (in any sequence) and connected serially by links whose cost is equal to $C_1$. This formation constitutes a line connecting the intermediate nodes to the two extreme nodes selected. Next, we consider the $G_2$ set consisting of the groups $G_2^{(1)},\ldots,G_2^{(g_2)}$. The nodes contained in a typical such group $G_2^{(m)}(1 \le m \le g_2)$ are ordered (in any sequence) and connected serially by links whose cost is equal to $C_2$. Nodes belonging to a $G_1^{(j)}$ group are considered as a single entity and are connected to their neighboring nodes through the two extreme nodes. This formation constitutes a spanning line connecting the nodes considered. This procedure is applied repeatedly until all the nodes are connected. Fig. 3 shows the spanning line corresponding to the group evolution process depicted in Fig. 2.

## 4. Complex node representation

Let $R(M_N(C))$ be a complex node representation corresponding to the matrix $M_N(C)$. Let $a_k$ denote the cost of the spoke associated with the node $n_k$, and $b_{i,j}$ the cost of the bypass associated with the pair of nodes $(n_i,n_j)$ as shown in Fig. 4. In order to reduce the path computation time in a homogeneous environment, Iliadis [4] considered the class of restricted optimal path complex node representations for which it holds that the optimal path can be either the direct exception bypass (if it exists), or the path through

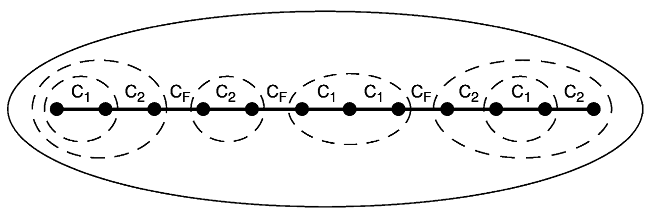



Fig. 3. Spanning line representation.

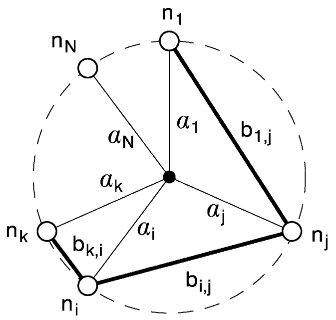



Fig. 4. Complex node representation.

the nucleus, i.e.

$$c_{i,j} = \begin{cases} \min(b_{i,j}, \max(a_i, a_j)) & \text{if } b_{i,j} \text{ exists,} \\ \max(a_i, a_j) & \text{otherwise,} \end{cases} \quad \forall i,j(1 \le i \ne j \le N).$$

In the present paper, the above restriction is removed. From the definition of the transition matrix it follows that the cost $c_{i,j}$ is the minimum of the costs of all possible paths connecting nodes $n_i$ and $n_j$. Therefore, in order to find the cost $c_{i,j}$, a search of all possible paths connecting nodes $n_i$ and $n_j$ should be conducted. In particular, considering the direct path through the exception bypass (if it exists), and the path through the nucleus, yields

$$c_{i,j} \le \begin{cases} \min(b_{i,j}, \max(a_i, a_j)) & \text{if } b_{i,j} \text{ exists,} \\ \max(a_i, a_j) & \text{otherwise,} \end{cases} \quad \forall i,j(1 \le i \ne j \le N). \tag{5}$$

Let $B(R)$ denote the number of exception bypasses used by the complex node representation $R$. Note that there may be several complex node representations corresponding to a given cost matrix. One complex node representation, for example, could be the following:

$$R_{\max}: \ a_i = \infty, \quad \forall i(1 \le i \le N)$$

$$\text{and } b_{i,j} = c_{i,j}, \quad \forall i,j(1 \le i \ne j \le N).$$

This representation uses exception bypasses for all pairs of nodes. Consequently, the number of exception bypasses used is the maximum possible and is equal to $B(R_{\max}) = N(N-1)/2$. Another complex node representation with a smaller number of exception bypasses is obtained by making use of the spanning line representation. Exception bypasses are used to connect only the neighboring nodes $j_k$ and $j_{k+1}$ on the line:

$$R_{\text{s\_line}}: \ a_i = \infty, \quad \forall i(1 \le i \le N)$$

$$\text{and } b_{j_k j_{k+1}} = c_{j_k j_{k+1}}, \quad \forall k(1 \le k \le N-1).$$

Consequently, the number of exception bypasses used is equal to $B(R_{\text{s\_line}}) = N - 1$. Our aim is to obtain a representation with a reduced number of exception bypasses.

As mentioned previously, the restricted optimal path complex node representations have the property that a restricted set of paths in the representation, specified a priori, always contains an optimal path. The set of paths was restricted to two paths, either a one-hop path using an exception bypass or a two-hop path obtained by the concatenation of two spokes. Removing this restriction on optimal paths, it potentially decreases the number of exception bypasses required by an exact optimal complex node representation as the set of complex node representations increases. This issue is addressed in Section 5.

## 5. Optimal complex node representations

In this section we establish the optimal substructure property of the optimal complex node representations, expressed by Theorem 2, which is key to deriving the method for obtaining them. We show that, as in the restricted case, in the unrestricted case the optimal substructure nature of the optimal complex node representations is also closely coupled with the group evolution process.

Let $S_R$ be the set of all possible complex node representations $R$ associated with the cost matrix $M_N(C)$. Our aim is to find the set of complex node representations $R_{\min}$ that use the minimum possible number of exception bypasses, as well as to determine this number denoted by $B_{\min}$. Thus,

$$B_{\min} = \min_{R \in S_R} \{B(R)\}, \tag{6}$$

and

$$B(R_{\min}) = B_{\min}. \tag{7}$$

Clearly,

$$0 \le B_{\min} \le B(R_{\text{s\_line}}) = N - 1 \le B(R_{\max}) = \frac{N(N-1)}{2}. \tag{8}$$

Furthermore, given that the class considered in Ref. [4] is a subset of the set of all possible complex node representations, the derived minimum number $B_{\min}^{\text{restricted}}$ in Ref. [4] constitutes an upper bound on the minimum number $B_{\min}$.

First, the properties of an optimal complex representation are identified by the following lemmas.

**Lemma 1.** *In an optimal complex node representation $R_{\min}$, for all pairs of nodes $(n_i, n_j)$ for which an exception bypass exists, it holds that*

$$c_{i,j} = b_{i,j} \qquad \text{and } c_{i,j} < \max(a_i, a_j). \tag{9}$$

**Proof.** See Appendix A. □

**Remark 1.** Note that in the optimal complex node representations, if an exception bypass exists, then it constitutes an optimal path between the corresponding adjacent nodes.

**Lemma 2.** *In an optimal complex node representation $R_{\min}$, there exists node $n_i$ such that*

$$a_i \le c_{\max} = C_F. \tag{10}$$

**Proof.** See Appendix A. □

Next we will show that the structure of an optimal complex node representation is closely coupled with the group evolution process. Let us consider the $k$th iteration of the process in which groups corresponding to the cost $C_k$ are formed. Let us focus on one typical group $G$ belonging to the set $G_k$, and let us assume that it contains the groups $S_1, \ldots, S_m, \ldots, S_Q$, as shown in Fig. 5. We will demonstrate that in order to obtain the optimal complex node representation of $G$, knowledge of the optimal complex node representations of the groups $S_1, \ldots, S_m, \ldots, S_Q$ is required.

Let us introduce the following definitions:

$R_{\min}(G)$ an optimal complex node representation corresponding to the nodes contained in $G$,
$B_{\min}(G)$ the number of exception bypasses used in $R_{\min}(G)$,
$|S_m|$ the number of nodes contained in the group $S_m$,
$R_{\min}(S_m)$ an optimal complex node representation corresponding to the nodes contained in $S_m$,
$B_{\min}(S_m)$ the number of exception bypasses used in $R_{\min}(S_m)$.

Based on the properties (3) and (4) of the group evolution process, the following two lemmas can be established:

**Lemma 3.** *For every pair of nodes $(n_i, n_j)$ belonging to the same group $S_m$, it holds that $c_{i,j} < C_k$.*

**Lemma 4.** *For every pair of nodes $(n_i, n_j)$ belonging to two different groups of $G$ (that is $n_i \in S_m$, $n_j \in S_f$, $m \neq f$), it holds that $c_{i,j} = C_k$.*

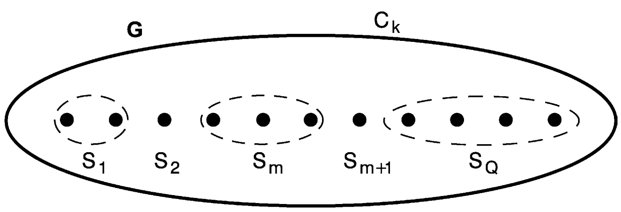



Fig. 5. Optimal complex node representation.

Next we explore the structure of $R_{\min}(G)$ with the following theorems.

**Theorem 1.** *There is no exception bypass between any pair of nodes belonging to two different groups.*

**Proof.** See Appendix A. □

**Theorem 2.** *In $R_{\min}(G)$,*

(*a*) *There exist exactly $Q-1$ groups, denoted by $S_1, \ldots, S_{m-1}, S_{m+1}, \ldots, S_Q$, having the following properties. In each one of these groups, there exists at least one node for which the cost of its spoke is equal to $C_k$, whereas the cost of the spokes of the remaining nodes of the group is at least $C_k$. Furthermore, the complex node representation corresponding to the nodes of group $S_m$ is an optimal one.*
(*b*) *The nodes within each of the $Q-1$ groups are connected by exception bypasses so as to form spanning trees or spanning lines.*
(*c*) *It holds that*

$$B_{\min}(G) = \min_{1 \le j \le Q} \left\{ B_{\min}(S_j) + \sum_{\substack{i=1 \\ i \ne j}}^{Q} g(|S_i|) \right\}, \tag{11}$$

*where*

$$g(x) \triangleq x - 1, \tag{12}$$

*and the group $S_m$ is identified by the following relation*:

$$B_{\min}(S_m) + \sum_{\substack{i=1 \\ i \ne m}}^{Q} g(|S_i|) = B_{\min}(G). \tag{13}$$

*If more than one group satisfies Eq. (13), group $S_m$ can be either one of them.*
(*d*) *For every pair of nodes $(n_i, n_j)$, there exists an optimal path connecting them containing at most two spokes.*

**Proof.** See Appendix A. □

**Remark 2.** The PNNI specification requires that a path through a logical node be obtained from a concatenation of any number of exception bypasses and at most two spokes in the complex node representation [1]. From Theorem 2, part (d), it follows that the optimal complex node representations satisfy this requirement.

### *5.1. Method for generating the optimal complex node representations*

From Theorem 2 it follows that in order to obtain the optimal set of complex node representations $R_{\min}(G)$, knowledge of the quantities $R_{\min}(S_m)$ and $B_{\min}(S_m)$ corresponding to each one of the groups $S_1, \ldots, S_m, \ldots, S_Q$ contained in the group $G$ is required.

*5.1.1. Algorithm for deriving the set of optimal complex node representations $R_{\min}(G)$ description*

a. The minimum number of exception bypasses corresponding to the optimal set of complex node representations is given by

$$B_{\min}(G) = \min_{1 \le j \le Q} \left\{ B_{\min}(S_j) + \sum_{\substack{i=1 \\ i \ne j}}^{Q} (|S_i| - 1) \right\}.$$

b. Let $S_m$ be a group (there exists at least one such group) that satisfies the following relation:

$$B_{\min}(S_m) + \sum_{\substack{i=1 \\ i \ne m}}^{Q} (|S_i| - 1) = B_{\min}(G).$$

c. Set the cost of the spokes corresponding to the nodes contained in the remaining $Q-1$ groups, $S_1, \ldots, S_{m-1}, S_{m+1}, \ldots, S_Q$, in a way such that, in each one of these groups, there exists at least one node for which the cost of its spoke is equal to $C_k$, whereas the cost of the spokes of the remaining nodes of the group is at least $C_k$.
d. Connect the nodes within each of the $Q-1$ groups by exception bypasses so as to form spanning trees or spanning lines.
e. Transfer the optimal complex representation of the group $S_m$, comprised of spokes and bypasses, onto the corresponding component of $R_{\min}(G)$.

The set of optimal complex node representations corresponding to the cost matrix $M_N(C)$ is obtained as follows. Starting at the lowest level, we follow the group evolution process based on the sorted cost entries $c_{\min} = C_1 < C_2 < \cdots < C_F = c_{\max}$. In a typical step $k$, the node groups $G_k^{(1)}, \ldots, G_k^{(g_k)}$ related to the cost $C_k$ are identified, and the corresponding set of optimal complex node representations is constructed by applying the above algorithm. In the final step, the optimal complex node representation sought is obtained, corresponding to the last group that contains all the nodes.

## 6. Numerical example

Let us consider the following cost matrix $M_7(C)$ also considered in Ref. [4].

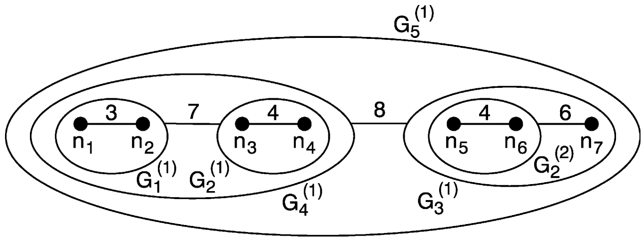



Fig. 6. Numerical example (group evolution process).

In this case we have: $c_{\min} = C_1 = 3$, $C_2 = 4$, $C_3 = 6$, $C_4 = 7$, and $c_{\max} = C_5 = 8$.

The group evolution process corresponding to this matrix is schematically shown in Fig. 6. Fig. 7 depicts the $R_{\text{s\_line}}$ complex node representation obtained by making use of the spanning line representation as discussed in Section 4.

$$M_7(C) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 3 & 7 & 7 & 8 & 8 & 8 \\ 3 & 0 & 7 & 7 & 8 & 8 & 8 \\ 7 & 7 & 0 & 4 & 8 & 8 & 8 \\ 7 & 7 & 4 & 0 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 0 & 4 & 6 \\ 8 & 8 & 8 & 8 & 4 & 0 & 6 \\ 8 & 8 & 8 & 8 & 6 & 6 & 0 \end{bmatrix} \end{array}$$

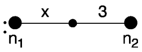
An optimal complex node representation is obtained in the following steps by applying the method described in Section 5.

Step 1:

$C_1 = 3$,

$G_1^{(1)} = \{n_1, n_2\}$,

$B_{\min}(G_1^{(1)}) = 0$,

$R_{\min}(G_1^{(1)})$ : 

$(x \le 3)$.

Step 2:

$C_2 = 4$,

$G_2^{(1)} = \{n_3, n_4\}$,

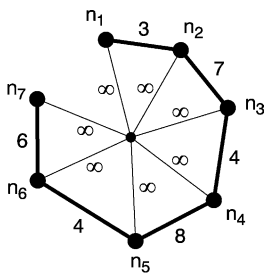
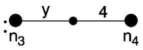



Fig. 7. $R_{\text{s\_line}}$ complex node representation.

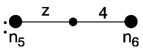
$B_{\min}(G_2^{(1)}) = 0,$

$R_{\min}(G_2^{(1)})$ : 

$(y \leq 4).$

$G_2^{(2)} = \{n_5, n_6\},$

$B_{\min}(G_2^{(2)}) = 0,$

$R_{\min}(G_2^{(2)})$ : 

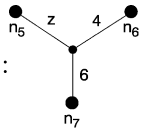
$(z \leq 4).$

Step 3:

$C_3 = 6,$

$G_3^{(1)} = G_2^{(2)} \cup \{n_7\} = \{n_5, n_6, n_7\},$

$B_{\min}(G_3^{(1)}) = \min\{B_{\min}(G_2^{(2)}) + 0, 0 + 1\} = 0,$

$R_{\min}(G_3^{(1)})$ : 

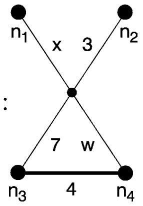
$(z \leq 4).$

Step 4:

$C_4 = 7,$

$G_4^{(1)} = G_1^{(1)} \cup G_2^{(1)} = \{n_1, n_2, n_3, n_4\},$

$B_{\min}(G_4^{(1)}) = \min\{B_{\min}(G_1^{(1)}) + 1, B_{\min}(G_2^{(1)}) + 1\} = 1,$

$R_{\min}(G_4^{(1)})$ : 

$(x \leq 3)$
$(w \geq 7)$.

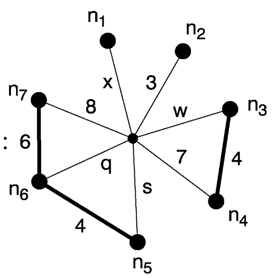
Step 5:

$C_5 = 8,$

$G_5^{(1)} = G_3^{(1)} \cup G_4^{(1)} = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\},$

$B_{\min}(G_5^{(1)}) = \min\{B_{\min}(G_3^{(1)}) + (4 - 1), B_{\min}(G_4^{(1)}) + (3 - 1)\}$

$= \min\{0 + 3, 1 + 2\} = 3,$

$R_{\min}(G_5^{(1)})$ : 

$(x \leq 3)$
$(w \geq 7)$
$(q \geq 8)$
$(s \geq 8)$.

The obtained optimal complex node representation uses three exception bypasses. Note that the representation obtained actually represents a set of optimal complex node representations because the $n_1$, $n_3$, $n_5$, and $n_6$ spokes can assume a range of values. Note also that in step 4, the exception bypass used to connect nodes $n_3$ and $n_4$ could instead have been used to connect nodes $n_1$ and $n_2$, resulting in another set of optimal complex node representations.

**Remark 3.** Comparing the optimal complex node representation obtained above with that obtained in Ref. [4] reveals that the representation obtained in Ref. [4] uses an additional exception bypass to connect nodes $n_5$ and $n_7$ because of the restriction imposed on optimal paths. As expected, removing the restriction on optimal paths results in an optimal complex node representation with fewer exception bypasses (three instead of four). The exception bypass between nodes $n_5$ and $n_7$ is no longer needed as the optimal cost of 6 is derived from the path $n_5 \rightarrow n_6 \rightarrow n_7$.

## 7. Bounds on the number of exception bypasses

In this section we derive the bounds on the number of exception bypasses used by the optimal complex node representations. We also determine the cost matrices that result in optimal complex node representations with a number of bypasses equal to these bounds.

In Section 6 we have shown how to construct an optimal complex representation $R_{\min}(M_N(C))$ corresponding to the cost matrix $M_N(C)$ using the least possible number of exception bypasses denoted by $B(R_{\min}(M_N(C)))$. Let us now consider all the possible cost matrices corresponding to $N$ border nodes and define:

$$a(N) \triangleq \min_{M_N(C)} \{B(R_{\min}(M_N(C)))\},$$
$$\text{and } b(N) \triangleq \max_{M_N(C)} \{B(R_{\min}(M_N(C)))\}. \tag{14}$$

From Eq. (8) it follows that both of these quantities exist and it holds that

$$0 \leq a(N) \leq B(R_{\min}(M_N(C))) \leq b(N)$$
$$\leq N - 1, \quad \forall M_N(C). \tag{15}$$

A cost matrix $M_N(C)$ is called *minimal* if $B(R_{\min}(M_N(C))) = a(N)$. Similarly, a cost matrix $M_N(C)$ is called *maximal* if $B(R_{\min}(M_N(C))) = b(N)$. Let $\{M_N(C_{\min})\}$ and $\{M_N(C_{\max})\}$ denote the sets of the minimal and maximal cost matrices, respectively.

### 7.1. Lower bound

**Theorem 3.** *It holds that*

$$a(N) = 0, \quad \forall N (N \geq 2). \tag{16}$$

*The group evolution process corresponding to the cost matrix $M_N(C_{\min})$ is shown in Fig. 8.*

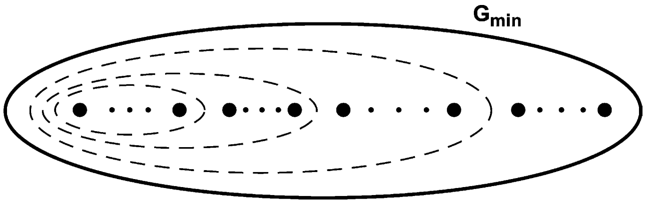



Fig. 8. Group evolution process corresponding to the cost matrix $M_N(C_{\min})$.

**Proof.** The proof is identical to that given in Section VIII.A of Ref. [4] and is therefore omitted. Also note that although in Ref. [4] the function $g(x)$ is differently defined, the proof still holds because the equation $g(x) = 0$ has the same root $x = 1$ in both cases. □

*7.2. Upper bound*

Let $M_N(C_{\max})$ be a maximal cost matrix so that $B(R_{\min}(M_N(C_{\max}))) = b(N)$. In this section we determine the value of $b(N)$, given by Theorem 7, and identify the structure of the corresponding group evolution process. We begin by defining the function

$$h(N) \triangleq g(N) - b(N), \quad \forall N. \tag{17}$$

Some basic properties of the functions $h(N)$ and $b(N)$ are established in Appendix B.

Let us now consider the last iteration of the process in which groups corresponding to the cost $c_{\max}$ are formed. According to the group evolution process, the last group $G$ containing all the $N$ nodes consists, in general, of $Q$ groups denoted by $S_1, \ldots, S_Q$, as shown in Fig. 5. Let $\overline{S_Q}$ represent the group evolution set $\{S_1, \ldots, S_Q\}$, with $|\overline{S_Q}| \triangleq \sum_{i=1}^{Q} |S_i| = N$, assuming, without loss of generality, that

$$|S_1| \leq |S_2| \leq \cdots \leq |S_Q|. \tag{18}$$

Let $\{M_N(C(\overline{S_Q}))\}$ represent the set of cost matrices that result in the above group evolution set. From Ref. [4] it holds that

$$b(N) = \max_{\overline{S_Q}} \{b(N|\overline{S_Q})\}, \quad \text{for } |\overline{S_Q}| = N, \tag{19}$$

with

$$b(N|\overline{S_Q}) \triangleq \max_{M_N \in \{M_N(C(\overline{S_Q}))\}} \{B_{\min}(M_N)\}. \tag{20}$$

A set $\overline{S_{Q^*}}$ is a *maximal group evolution set* if and only if $b(N|\overline{S_{Q^*}}) = b(N)$. Also $\{M_N(C_{\max}(\overline{S_Q}))\}$ denotes the subset of $\{M_N(C(\overline{S_Q}))\}$ containing the cost matrices $M_N$ for which it holds that $B_{\min}(M_N) = b(N|\overline{S_Q})$. We proceed by establishing the following theorems based upon which the result sought is obtained. Their proof is given in Appendix B.

**Theorem 4.** *For a group evolution set $\overline{S_Q}$ with $|S_1| \leq |S_2| \leq \cdots \leq |S_Q|$, it holds that*

$$b(N|\overline{S_Q}) = b(|S_Q|) + \sum_{i=1}^{Q-1} g(|S_i|). \tag{21}$$

*A cost matrix $M_N$ belongs to the set $\{M_N(C_{\max}(\overline{S_Q}))\}$ if and only if it satisfies the following conditions*:

$$B_{\min}(S_Q(M_N)) = b(|S_Q|), \tag{22}$$

*and*

$$B_{\min}(S_j(M_N)) \geq b(|S_Q|) - g(|S_Q|) + g(|S_j|), \quad \forall j, \ 1 \leq j \leq Q - 1, \tag{23}$$

*with $S_j(M_N)$ denoting the cost matrix corresponding to the nodes contained in group $S_j$.*

**Proof.** Immediate from Remark 2 of Ref. [4] and Lemma 6 of Appendix B. □

**Remark 4.** Note that for $|S_j| < |S_Q|$ and by virtue of Eqs. (14), (17) and (B6), Eq. (23) yields $b(|S_j|) \geq B_{\min}(S_j(M_N)) \geq b(|S_Q|) - g(|S_Q|) + g(|S_j|)$.

**Remark 5.** Note that if a cost matrix $M_N$ ($M_N \in \{M_N(C(\overline{S_Q}))\}$) satisfies the following conditions $B_{\min}(S_j(M_N(C(\overline{S_Q})))) = b(|S_j|)$, $\forall j$, $1 \leq j \leq Q$, by virtue of Remark 4, it also satisfies the conditions (22) and (23) and it, therefore, belongs to the set $\{M_N(C_{\max}(\overline{S_Q}))\}$.

**Corollary 1.** *For a maximal group evolution set $\overline{S_{Q^*}}$ with $|S_1| \leq |S_2| \leq \cdots \leq |S_{Q^*}|$, it holds that*

$$b(N) = b(N|\overline{S_{Q^*}}) = b(S_{Q^*}) + \sum_{i=1}^{Q^*-1} g(|S_i|). \tag{24}$$

**Proof.** Immediate from Eq. (21). □

**Lemma 5.** *The sequence $b(N)$ is increasing in $N$, i.e. $b(N-1) \leq b(N)$, $\forall N$.*

**Proof.** The proof is identical to the proof of Lemma 10 in Ref. [4] and is therefore omitted. □

Next we establish the following theorems based upon which the result sought is obtained.

**Theorem 5.** *For a maximal group evolution set $\overline{S_{Q^*}}$, it holds that $Q^* \leq 3$.*

**Proof.** The proof is identical to the proof of Theorem 6 in Ref. [4] and is therefore omitted. Note that this proof uses Lemmas 11 and 12, both of which hold in our case. The proof of Lemma 12 remains the same, whereas the proof of Lemma 11 needs to be modified by considering the new function $g(x)$ and replacing the last term $b(N) + |S_1||S_2|$ with the term $b(N) + 1$. □

Table 1
Upper bound values $b(N)$

| $N$ | $b(n)$ | $N$ | $b(n)$ | $N$ | $b(n)$ | $N$ | $b(n)$ | $N$ | $b(n)$ | $N$ | $b(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 11 | 6 | 21 | 15 | 31 | 25 | 41 | 34 | 51 | 44 |
| 2 | 0 | 12 | 7 | 22 | 16 | 32 | 26 | 42 | 35 | 52 | 45 |
| 3 | 0 | 13 | 8 | 23 | 17 | 33 | 26 | 43 | 36 | 53 | 46 |
| 4 | 1 | 14 | 9 | 24 | 18 | 34 | 27 | 44 | 37 | 54 | 47 |
| 5 | 1 | 15 | 10 | 25 | 19 | 35 | 28 | 45 | 38 | 55 | 48 |
| 6 | 2 | 16 | 11 | 26 | 20 | 36 | 29 | 46 | 39 | 56 | 49 |
| 7 | 3 | 17 | 11 | 27 | 21 | 37 | 30 | 47 | 40 | 57 | 50 |
| 8 | 4 | 18 | 12 | 28 | 22 | 38 | 31 | 48 | 41 | 58 | 51 |
| 9 | 4 | 19 | 13 | 29 | 23 | 39 | 32 | 49 | 42 | 59 | 52 |
| 10 | 5 | 20 | 14 | 30 | 24 | 40 | 33 | 50 | 43 | 60 | 53 |

**Theorem 6.** *There always exists a maximal group evolution set* $\overline{S_{Q^*}}$, *with* $Q^* = 2$.

**Proof.** The proof is identical to the proof of Theorem 7 in Ref. [4] and is therefore omitted. □

**Theorem 7.** *It holds that*

$$b(N) = N - 1 - \lceil \log_2 N \rceil, \quad \text{with } b(N-k) + g(k) = \begin{cases} b(N) - 1, & 1 \le k < N - 2^{\lfloor \log_2 (N-1) \rfloor}, \\ b(N), & N - 2^{\lfloor \log_2 (N-1) \rfloor} \le k \le \left\lfloor \dfrac{N}{2} \right\rfloor. \end{cases} \tag{25}$$

**Proof.** The proof is given in Appendix B. □

The values of the function $b(N)$ for $N \le 60$ are listed in Table 1.

**Corollary 2.** *A set* $\overline{S_2}$ *with* $|\overline{S_2}| = N$ *and* $|S_1| \le |S_2|$ *is a maximal group evolution set if and only if* $N - 2^{\lfloor \log_2 (N-1) \rfloor} \le |S_1| \le \lfloor N/2 \rfloor$.

**Proof.** Immediate from Eqs. (21), (24) and (25). □

**Remark 6.** In Ref. [4] it was shown that, in the case of restricted optimal path complex node representations, the upper bound on the number of exception bypasses grows quadratically in the number of border nodes. Removing the restriction on optimal paths results in an upper bound, which grows linearly in the number of nodes. In the case of a heterogeneous environment, this results in a decreased path computation time compared with the time required when the restricted optimal path complex node representations are deployed.

Next we shall show that the group evolution process shown in Fig. 9 corresponds to a maximal cost matrix $M_N \in \{M_N(C_{\max})\}$. According to Corollary 2, the choice $|S_1| = \lfloor N/2 \rfloor$ and $|S_2| = \lceil N/2 \rceil$ corresponds to a maximal group evolution set. According to Remark 5, for the cost matrix $M_N$ to be maximal it suffices the cost matrices corresponding to the two groups $S_1$ and $S_2$ to also be maximal. This is ensured by the nested form of the group evolution process owing to the repeated application of this choice.

The proof of the following theorems is given in Appendix B.

**Theorem 8.** *The equation* $b(N) = b(N+1)$ *is satisfied if and only if* $N = 2^k$, $k = 0, 1, \ldots$.

**Theorem 9.** *A maximal group evolution set* $\overline{S_3}$ *exists if and only if* $N$ *lies in the range* $2^k + 1 \le N \le 3 \cdot 2^{k-1}$, $k = 1, 2, \ldots$.

**Corollary 3.** *By using the optimal complex node representation instead of the* $R_{s\_line}$ *representation, the savings in terms of the number of exception bypasses used is between* $\lceil \log_2 N \rceil / (N-1)$ *and 100%.*

**Proof.** Immediate from Eqs. (15), (16) and (25). □

**Corollary 4.** *As* $N$ *increases, the number of exception bypasses required is negligible compared to the maximum possible, i.e.*

$$\lim_{N \to \infty} \frac{b(N)}{N(N-1)/2} = 0.$$

**Proof.** Immediate from Eq. (25). □

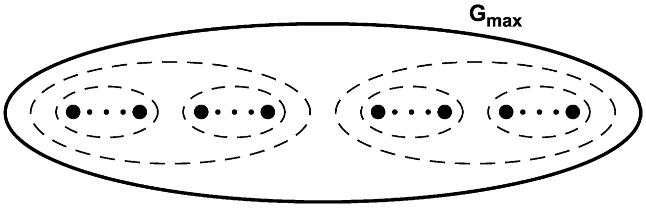



Fig. 9. A maximal group evolution set corresponding to the cost matrix $M_N(C_{\max})$ with $N = 2^k$.

## 8. Conclusions

The complex node representation is an important topic in the context of the PNNI protocol. It allows the cost of traversing the peer group to be advertised in a compact form. Complex node representations using a small number of links result in a reduced amount of path computation time and in improved performance. In this paper, we considered the case of an efficient deployment of complex node representations in a heterogeneous environment. The method for constructing the set of the optimal complex node representations for restrictive and symmetric costs was presented. This method yields representations that use the minimum possible number of links and, as it turns out, result in significant savings compared with the previously developed restricted optimal path complex node representations. The derivations of optimal complex node representations in the cases of asymmetric and additive costs are topics for further investigation.

## Appendix A. Properties of the optimal complex node representations

**Proof of Lemma 1.** For the purpose of contradiction, suppose that $c_{i,j} < b_{i,j}$. This implies that the optimal path from $n_i$ to $n_j$ does not include bypass $b_{i,j}$ and, therefore, this bypass can be deleted without affecting the cost. This, however, results in another representation with fewer bypasses, contradicting the assumption that $R_{\min}$ is optimal. Consequently, $c_{i,j} \geq b_{i,j}$ and from Eq. (5) it follows that $c_{i,j} = b_{i,j}$. Suppose now that $\max(a_i, a_j) = c_{i,j}$. This also leads to contradiction because it implies that the bypass $b_{i,j}$ can be deleted without affecting the cost. Consequently, $\max(a_i, a_j) > c_{i,j} = b_{i,j}$ and this completes the proof of Lemma 1. □

**Proof of Lemma 2.** Suppose on the contrary that $a_k > c_{\max}$ for all nodes $n_k$. Obviously, there exists bypass with $b_{i,j} = c_{\max}$, for otherwise $c_{i,j} \neq c_{\max}\ \forall(n_i, n_j)$. Note that reducing the cost of both spokes from $a_i$ and $a_j$ to $c_{\max}$ does not affect the cost between any pair of nodes and it therefore results in another optimal representation. Furthermore, $\max(a_i, a_j) = c_{\max}$, contradicting Eq. (9). □

**Proof of Theorem 1.** Suppose on the contrary that there exists exception bypass $b_{i,j}$ connecting nodes $n_i$ and $n_j$ with $n_i \in S_m$, $n_j \in S_f$, $m \neq f$. From Eq. (9) and Lemma 4, it follows that $C_k = c_{i,j} < \max(a_i, a_j)$. Let us assume, without loss of generality, that $a_i > C_k$. Note that reducing the cost of this spoke from $a_i$ to $C_k$, and, in the case where $a_j > C_k$, the cost of the spoke $n_j$ from $a_j$ to $C_k$, does not affect the cost between any pair of nodes and it therefore results in another optimal representation. Furthermore, it holds that $\max(a_i, a_j) = C_k = c_{i,j}$ contradicting Eq. (9). □

**Proof of Theorem 2.** (a) First we shall show that there can be at most one group that contains nodes for which the cost of spokes is less than $C_k$. Suppose on the contrary that there exist two groups $S_m$ and $S_f$ and nodes $n_i$ ($n_i \in S_m$) and $n_j$ ($n_j \in S_f$), such that the cost of the spokes associated with these nodes is less than $C_k$, i.e. $a_i < C_k$ and $a_j < C_k$. It now follows that $c_{i,j} \leq \max(a_i, a_j) < C_k$, which contradicts Lemma 4. We have shown that there exist at least $Q - 1$ groups, denoted by $S_1, \ldots, S_{m-1}, S_{m+1}, \ldots, S_Q$, for which the cost of the spokes of the nodes contained in these groups is at least $C_k$. For the purpose of contradiction, suppose that there exists among them group $S_f$ with the costs of all the spokes corresponding to its nodes being greater than $C_k$, and let us consider node $n_j$ belonging to this group and node $n_i$ belonging to a different group. From Theorem 1, it follows that all paths connecting $n_j$ to $n_i$ go over the nucleus and, therefore, contain a spoke whose cost exceeds $C_k$. This implies that $c_{i,j} > C_k$, contradicting Lemma 4. Consequently, in each of the $Q - 1$ groups, there exists at least one node for which the cost of its spoke is equal to $C_k$.
Next, we shall show that the complex node representation corresponding to the group $S_m$ is optimal. Suppose on the contrary that the complex node representation corresponding to the group $S_m$ is not optimal. Let us replace the component of $R_{\min}(G)$ corresponding to the group $S_m$ with the complex node representation $R_{\min}(S_m)$. From Eq. (10) and Lemma 3 it follows that there exists spoke whose value is less than $C_k$. Therefore, the resulting new representation is a valid complex representation of group $G$ with fewer exception bypasses than $R_{\min}(G)$. This contradicts the assumption of optimality of $R_{\min}(G)$. Because in representation $R_{\min}(S_m)$ there exists spoke whose value is less than $C_k$, the number of groups referred to above is exactly $Q - 1$.
(b) Let us consider the group $S_f$, $f \neq m$. From part (a) it follows that the values of its spokes are at least $C_k$, whereas from Lemma 3, it follows that $c_{i,j} < C_k$. Consequently, the optimal paths connecting the nodes of the group $S_f$ do not include any spokes but only exception bypasses. The most efficient structures to connect these nodes are those discussed in Section 3.2. Consequently, the total number of exception bypasses corresponding to the group $S_f$ is $g(|S_f|) = |S_f| - 1$.
(c) From parts (a) and (b) it follows that the total number of exception bypasses used in $R_{\min}(G)$ is equal to

$$B_{\min}(S_m) + \sum_{\substack{f=1 \\ f \neq m}}^{Q} g(|S_f|).$$

Note that there are $Q$ possible values for $m$. As the optimal complex representation must use one of these values for $m$, we only need to check them all to find the best one.
(d) Assuming that this proposition holds for each optimal complex node representation corresponding to each of the groups $S_1, \ldots, S_Q$ contained in the group $G$, we shall show that it also holds for the optimal complex node

representation $R_{\min}(G)$. The following cases are considered:

Case (1)
$n_i \in S_m$, and $n_j \in S_m$. From part (a) it follows that the complex node representation corresponding to the nodes of group $S_m$ is an optimal one. This, according to the above-mentioned assumption, implies that the proposition holds.
Case (2)
$n_i \in S_f$, and $n_j \in S_f$, $f \neq m$. From part (b) it follows that there exists optimal path obtained from a concatenation of only exception bypasses. Consequently, the proposition holds.
Case (3)
$n_i \in S_q$, and $n_j \in S_f$, $q \neq f$. From parts (a) and (b) it follows that an optimal path connecting nodes $n_i$ and $n_j$ exists obtained from a concatenation of exception bypasses, and two spokes corresponding to the groups $S_q$ and $S_f$, respectively, whose cost is equal to $C_k$. Consequently, the proposition holds.

□

## Appendix B. Upper bound on the number of exception bypasses

**Lemma 6.** *The sequence $h(N)$ is increasing in $N$,*

$$h(N-1) \leq h(N), \quad \forall N. \tag{B1}$$

**Proof.** Let us consider a cost matrix $M_N(C_{\max})$ such that $B(R_{\min}(M_N(C_{\max}))) = b(N)$, and let $c_{\max}$ denote the corresponding maximum cost entry. Let $M_{N-1}(C)$ be the cost matrix associated with the first $N-1$ nodes, and let $R_{\min}(M_{N-1}(C))$ be the corresponding optimal complex node representation. From definition (14) it follows that

$$B(R_{\min}(M_{N-1}(C))) \leq b(N-1). \tag{B2}$$

A complex node representation (not necessarily optimal) $R(M_N(C_{\max}))$ of the original cost matrix $M_N(C_{\max})$ is now constructed based on the $R_{\min}(M_{N-1}(C))$ using the following procedure. Firstly, we set $a_N$ equal to $\infty$. Considering the group evolution process, suppose that node $n_N$ appears for the first time at the $k$th iteration, implying, according to the properties of the group evolution process, that $c_{N,j} \geq C_k$ $\forall j$, $j \neq N$. Also let $n_i$ be a node such that $c_{N,i} = C_k$. We now introduce an exception bypass between nodes $n_N$ and $n_i$ with cost $b_{N,i} = c_{N,i} = C_k$. Clearly, for this representation it holds that $c_{N,j} \geq C_k$ $\forall j$, $1 \leq j \leq N-1$.
This is an accurate representation because

(a) For all nodes $n_j$ for which it holds that $c_{N,j} = C_k$, based on the properties of the group evolution process it follows that $c_{i,j} \leq C_k$. Therefore, the optimal path $n_i \overset{\mathscr{P}}{\rightsquigarrow} n_j$ of the $R_{\min}(M_{N-1}(C_{\max}))$ has cost not exceeding $C_k$. Furthermore, the cost $c_{N,j}$ based on $\underset{\mathscr{P}}{R}(M_N(C))$ is also equal to $C_k$ due to the path $n_N \rightarrow n_i \overset{\mathscr{P}}{\rightsquigarrow} n_j$.

(b) For the remainder of the nodes $n_j$ it holds that $c_{N,j} > C_k$, and from Eq. (4), it follows that $c_{N,j} = c_{i,j}$. Note that the optimal path $n_i \overset{\mathscr{P}}{\rightsquigarrow} n_j$ of the $R_{\min}(M_{N-1}(C_{\max}))$ has cost equal to $c_{i,j}$. Furthermore, the cost $c_{N,j}$ based on $\underset{\mathscr{P}}{R}(M_N(C))$ is also equal to $c_{i,j}$ due to the path $n_N \rightarrow n_i \overset{\mathscr{P}}{\rightsquigarrow} n_j$.

The total number of exception bypasses used by this representation is given by

$$B(R(M_N(C_{\max}))) = B(R_{\min}(M_{N-1}(C))) + 1. \tag{B3}$$

From the definitions given by Eqs. (6) and (7) it holds that

$$B(R(M_N(C_{\max}))) \geq B(R_{\min}(M_N(C_{\max}))) = b(N). \tag{B4}$$

Combining Eqs. (B2)–(B4) and using Eq. (12) gives

$$b(N) \leq b(N-1) + 1 = b(N-1) + g(N) - g(N-1)$$

which by virtue of Eq. (17) yields the result and this completes the proof of Lemma 6. □

**Proof of Theorem 7.** We shall prove the theorem using mathematical induction. For $N = 1, 2$ the theorem holds because $b(1) = b(2) = 0$. Suppose that Eq. (25) holds for all $N \leq L$, so that

$$b(N) = N - 1 - \lceil \log_2 N \rceil, \quad \forall N \leq L. \tag{B5}$$

We shall prove that Eq. (B5) is also true for $L+1$. To that end, it suffices to show that

$$b(L+1) = L - \lceil \log_2(L+1) \rceil, \text{ with } b(L+1-k) + g(k)$$
$$= \begin{cases} b(L+1) - 1, & 1 \leq k < L+1-2^{\lfloor \log_2 L \rfloor}, \\ b(L+1), & L+1-2^{\lfloor \log_2 L \rfloor} \leq k \leq \left\lfloor \dfrac{L+1}{2} \right\rfloor. \end{cases} \tag{B6}$$

Using Theorem 6 and Eq. (24), Eq. (19) yields

$$b(N) = \max_{\overline{S_2}} \{ b(N|\overline{S_2}) \} = \text{with } |S_1| \leq |S_2|, |S_1| + |S_2| = N, \ \forall N,$$

or

$$b(N) = \max_{1 \leq k \leq \lfloor N/2 \rfloor} \{ b(N-k) + g(k) \}, \ \forall N.$$

In particular, for $N = L+1$, and using Eqs. (12) and (B5) the above yields

$$b(L+1) = \max_{1 \leq k \leq \left\lfloor \frac{L+1}{2} \right\rfloor} \{ L - 1 - \lceil \log_2(L+1-k) \rceil \}. \tag{B7}$$

It can be shown that the following relations hold

$$\lceil \log_2(L+1) \rceil = \lfloor \log_2 L \rfloor + 1, \ \forall L \geq 1,$$

$$\text{and } 2^{m-1} < \left\lceil \frac{L+1}{2} \right\rceil \leq 2^m \leq L < 2^{m+1}, \text{ with } m \tag{B8}$$
$$= \lfloor \log_2 L \rfloor, \ \forall L \geq 2.$$

Hence,

$$1 \le k < L+1-2^m \Leftrightarrow 2^m < L+1-k \le L$$
$$\Rightarrow \lceil \log_2(L+1-k) \rceil = \lfloor \log_2 L \rfloor + 1,$$
$$\text{and } L+1-2^m \le k < \left\lfloor \frac{L+1}{2} \right\rfloor \Leftrightarrow \left\lceil \frac{L+1}{2} \right\rceil < L+1-k \le 2^m \Rightarrow \lceil \log_2(L+1-k) \rceil = \lfloor \log_2 L \rfloor. \quad \text{(B9)}$$

Consequently,

$$L-1-\lceil \log_2(L+1-k) \rceil = \begin{cases} L-2-\lfloor \log_2 L \rfloor, & 1 \le k < L+1-2^{\lfloor \log_2 L \rfloor}, \\ L-1-\lfloor \log_2 L \rfloor, & L+1-2^{\lfloor \log_2 L \rfloor} \le k \le \left\lfloor \frac{L+1}{2} \right\rfloor. \end{cases} \quad \text{(B10)}$$

The above yields Eq. (B6), given that using Eqs. (B7) and (B8) we get

$$b(L+1) = L-1-\lfloor \log_2 L \rfloor = L-(1+\lfloor \log_2 L \rfloor) = L - \lceil \log_2(L+1) \rceil.$$

□

**Proof of Theorem 8.** It holds that

$$b(N) = b(N+1) \Leftrightarrow N-1-\lceil \log_2 N \rceil = N - \lceil \log_2(N+1) \rceil \Leftrightarrow \lceil \log_2(N+1) \rceil = \lceil \log_2 N \rceil + 1.$$

By making use of Eq. (B8), the above is written

$$b(N) = b(N+1) \Leftrightarrow \lfloor \log_2 N \rfloor = \lceil \log_2 N \rceil \Leftrightarrow \log_2 N = k \Leftrightarrow N = 2^k, \quad k = 0, 1, \ldots$$

□

**Proof of Theorem 9.** Let the last group $G$ of the group evolution process contain $N$ nodes and consist of three groups, i.e. $\overline{S_3} = \{S_1, S_2, S_3\}$ with $|S_1| \le |S_2| \le |S_3|$. According to Lemma 12 of Ref. [4], the set $\overline{S_3}$ is a maximal group evolution set if and only if the sets $\overline{S_2'}$ and $\overline{S_2''}$ are maximal. Applying Corollary 2 to the sets $\overline{S_2'}$ and $\overline{S_2''}$ yields $N - 2^{\lfloor \log_2(N-1) \rfloor} \le |S_1'| = |S_1| \le \lfloor N/2 \rfloor$, and $N - |S_1| - 2^{\lfloor \log_2(N-|S_1|-1) \rfloor} \le |S_1''| = |S_2| \le \lfloor (N-|S_1|)/2 \rfloor$, respectively. From the above it follows that $N - 2^{\lfloor \log_2(N-1) \rfloor} \le |S_1| \le \lfloor N/3 \rfloor$. Note that for any value of $|S_1|$ chosen within the specified range, the choice $|S_2| = \lfloor (N-|S_1|)/2 \rfloor$ and $|S_3| = \lceil (N-|S_1|)/2 \rceil$ results in a maximal group evolution set. Consequently, a maximal group evolution set exists if and only if

$$1 \le N - 2^{\lfloor \log_2(N-1) \rfloor} \le \left\lfloor \frac{N}{3} \right\rfloor \Leftrightarrow 1 \le N - 2^{\lfloor \log_2(N-1) \rfloor} \le \frac{N}{3} \Leftrightarrow 2^{\lfloor \log_2(N-1) \rfloor} + 1 \le N \le 3 \cdot 2^{\lfloor \log_2(N-1) \rfloor - 1}.$$

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