Discrete Mathematical Structures
CS 2233 Lecture Two

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Course Introduction

- **Recall:** Homework one due Thursday 1/22
  - Section 1.1: 6d, 6e, 6g, 10, 26, 28a-d
  - Hand in at beginning of class. Work alone.
- Recitation will be held today, Thursday 1/15
  - I plan to use the Thursday and Tuesday Prior to a homework assignment being due to give you a chance to ask questions about it
  - If you find you want to attend a different recitation than the one you have registered for, that's OK

Syntax of Propositional Formulas

- Definition of *propositional formula:*
  - A propositional variable *p* is a propositional formula
  - The constants **T** and **F** are propositional formulas
  - If ϕ and ψ are propositional formulas, then the following are also propositional formulas:
    - (ϕ)
    - $\neg\phi$
    - $\phi \wedge \psi$
    - $\phi \vee \psi$
    - $\phi \rightarrow \psi$
    - $\phi \leftrightarrow \psi$
    - $\phi \oplus \psi$

Semantics of Propositional Formulas

- A formula defines a function from truth assignments to truth values
  - A truth assignment gives a truth value for each variable

Negation

- Def: Given a proposition p, ¬p denotes the negation of p
  - ¬p means "it is not the case that p"
- Truth table for ¬p:

p	¬p
T	F
F	T

Given truth assignments for p

Resulting truth value of ¬p

Conjunction

- Def: Given propositions p and q, p ∧ q denotes the conjunction of p and q
  - p ∧ q means "p and q "
- Truth table for p ∧ q:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Given truth assignments

Resulting truth value

Disjunction

- Def: Given propositions p and q, $p \lor q$ denotes the disjunction of p and q
  - $p \lor q$ means "p or q" (*inclusive or*)
- Truth table for $p \lor q$:

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

Given truth assignments — Resulting truth value

Implication

- Def: Given propositions p and q, $p \to q$ is an implication
  - $p \to q$ means "p implies q"
  - p is the *hypothesis, antecedent* or *premise*
  - q is the *conclusion, consequence,* or *consequent*
  - Truth table for $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Understanding Implications

- Some readings of $p \to q$:
  - if p, then q
  - q if p
  - q when p
  - p only if q
  - q follows from p
  - q is a necessary condition for p
  - p is a sufficient condition for q
  - q is necessary for p
  - p is sufficient for q

Formulas Related to $p \to q$

- *Contrapositive*: $\neg q \to \neg p$
- *Converse*: $q \to p$
- *Inverse*: $\neg p \to \neg q$
- Equivalent formula not using $\to$
  $\neg p \lor q$

Equivalence

- Two formulas are equivalent if they have the same semantics
  - They yield the same truth values on the same truth assignments

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$	$\neg p \lor q$
T	T	F	F	T	T	T	T	T
T	F	F	T	F	F	T	T	F
F	T	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T	T

Notation for Equivalence

- When ϕ and ψ are are we write $\phi \equiv \psi$
  - e.g., $(\phi \to \psi) \equiv (\neg\phi \lor \psi)$
- Note that $\equiv$ is not a logical connective (*i.e.,* a logical operator)
  - These are not formulas:
    - $\phi \equiv \psi$
    - $(\phi \equiv \psi) \land p$
- $\equiv$ is part of the *meta-language* we use for discussing formulas in the *object language* of propositional calculus

Exclusive Or

- Def: Given propositions p and q, p ⊕ q denotes the exclusive or of p and q
  - p ⊕ q means "p or q, but not both"
- Truth table for p ⊕ q:

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Biconditionals

- Def: Given propositions p and q, p ↔ q is a *biconditional*
  - p ↔ q means "p if and only if q"
- Truth table for p ↔ q:

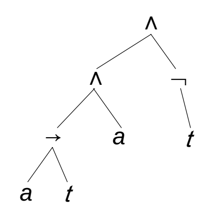
p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Translating English to Logic

- Fred can access the wireless network only if Fred has paid his tuition
  - Let *a* represent "Fred can access wireless"
  - Let *t* represent "Fred has paid his tuition"
  - *a* → *t*

Expression Trees

- Example: ((*a* → *t*) ∧ *a*) ∧ ¬*t*



A More Concise Truth Table

((*a*	→	*t*)	∧	*a*)	∧	¬	*t*
T		T		T			T
T		F		T			F
F		T		F			T
F		F		F			F

A More Concise Truth Table

((*a*	→	*t*)	∧	*a*)	∧	¬	*t*
T	T	T		T		F	T
T	F	F		T		T	F
F	T	T		F		F	T
F	T	F		F		T	F

A More Concise Truth Table

$((a$	$\rightarrow$	$t\,)$	$\wedge$	$a)$	$\wedge$	$\neg$	t
T	T	T	T	T		F	T
T	F	F	F	T		T	F
F	T	T	F	F		F	T
F	T	F	F	F		T	F

A More Concise Truth Table

$((a$	$\rightarrow$	$t\,)$	$\wedge$	$a)$	$\wedge$	$\neg$	t
T	T	T	T	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	F	F	F	F	T
F	T	F	F	F	F	T	F

Precedence

- Should $\neg q \rightarrow \neg p$ be interpreted as
  - $(\neg q) \rightarrow (\neg p)$, or as
  - $\neg(q \rightarrow (\neg p))$?
- Precedence gives rules for implicit parentheses

Op	Prec
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Consistency

- Intuitive requirement: specifications should not contain conflicting requirements
- Definition: A collection of propositional formulas is *consistent* if there is a truth assignment that makes each formula true

Example Inconsistent Spec

- Specification:
  - Fred can access the wireless network only if Fred has paid his tuition $(a \rightarrow t)$
  - Fred can access the wireless network (a)
  - Fred has not paid his tuition $(\neg t)$

a	t	$a \rightarrow t$	$\neg t$	$(a \rightarrow t) \wedge a \wedge \neg t$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

- $(a \rightarrow t) \wedge a \wedge \neg t$ is a *contradiction*

Tautologies and Contradictions

- A compound proposition that is true for all truth assignments is a *tautology*
  - *E.g.*, $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
- One that is false for all assignments is a *contradiction*