Discrete Mathematical Structures

CS 2233 Lecture Three

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Business

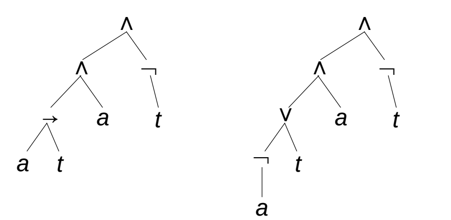
- Homework 2 due Thursday January 29
  – Section 1.2: 2, 6, 10
  – Section 1.3: 10d, 10e, 14, 24c, 24d, 32a, 32b, 44
- **Recall:** Homework 1 due Thursday January 22
  – Section 1.1: 6d, 6e, 6g, 10, 26, 28a-d
  – Hand in at beginning of class. Work alone.
- Questions???

Deriving Equivalence

- Recall: Two formulas are equivalent if they have the same semantics
  – They yield the same truth values on the same truth assignments
- Problem: Given two formulas, show they are equivalent
- Two methods
  1. Construct truth tables for both formulas
  2. Use known equivalences to derive equivalence of the given formulas by using two rules:
     1. Instantiation
     2. Substitution
  3. An additional method that should be listed here and was illustrated in class:
     – Use a sequence of equivalences to rewrite one formula into the other

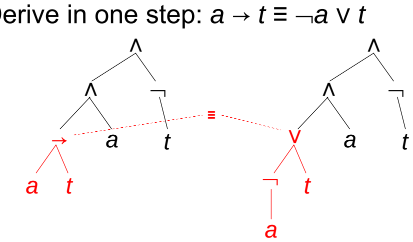
Deriving Equivalence: Example

- Recall: $(\phi \rightarrow \psi) \equiv (\neg\phi \vee \psi)$
- Derive: $((a \rightarrow t) \wedge a) \wedge \neg t \equiv ((\neg a \vee t) \wedge a) \wedge \neg t$



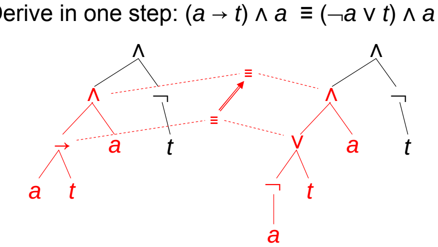
Deriving Equivalence: Instantiation Step

- Use: $(\phi \rightarrow \psi) \equiv (\neg\phi \vee \psi)$ with $\phi = a$ and $\psi = t$
- Derive in one step: $a \rightarrow t \equiv \neg a \vee t$



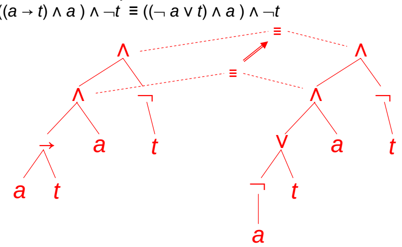
Substitution Step

- Use: $a \rightarrow t \equiv \neg a \vee t$ and $a \equiv a$
- Derive in one step: $(a \rightarrow t) \wedge a \equiv (\neg a \vee t) \wedge a$



Substitution Step

- Use:
  - $(a \rightarrow t) \wedge a \equiv (\neg a \vee t) \wedge a$
  - $\neg t \equiv \neg t$
- Derive in one step:
  - $((a \rightarrow t) \wedge a) \wedge \neg t \equiv ((\neg a \vee t) \wedge a) \wedge \neg t$



Substitution Rule

- Suppose $\phi_1 \equiv \phi_2$ and $\psi_1 \equiv \psi_2$
  - Then for any logical connective μ, $\phi_1\ \mu\ \psi_1 \equiv \phi_2\ \mu\ \psi_2$
  - Also $\neg\phi_1 \equiv \neg\phi_2$

Some Important Logical Equivalences

$\phi \wedge T \equiv \phi$ $\phi \vee F \equiv \phi$	Identity laws
$\phi \vee T \equiv T$ $\phi \wedge F \equiv F$	Domination laws
$\phi \vee \phi \equiv \phi$ $\phi \wedge \phi \equiv \phi$	Idempotency laws
$\neg(\neg\phi) \equiv \phi$	Double nagation law
$\phi \vee \psi \equiv \psi \vee \phi$ $\phi \wedge \psi \equiv \psi \wedge \phi$	Commuitivity laws

More Equivalences

$(\phi \vee \psi) \vee \theta \equiv \phi \vee (\psi \vee \theta)$ $(\phi \wedge \psi) \wedge \theta \equiv \psi \wedge (\phi \wedge \theta)$	Associativity laws
$\phi \vee (\psi \wedge \theta) \equiv (\phi \vee \psi) \wedge (\phi \vee \theta)$ $\phi \wedge (\psi \vee \theta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \theta)$	Distributivity laws
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$	De Morgan's laws
$\phi \vee (\phi \wedge \psi) \equiv \phi$ $\phi \wedge (\phi \vee \psi) \equiv \phi$	Absorption laws
$\phi \vee \neg\phi \equiv T$ $\phi \wedge \neg\phi \equiv F$	Negation laws

Exercise

- Use known equivalences to show the following:

  $\neg(p \rightarrow q) \equiv p \wedge \neg q$
- It was shown in class how to rewrite the left-hand formula into the right-hand formula by using a sequence of equivalences

Section 1.3: Predicates

- Propositional functions
  - A *propositional function* p and a *predicate* p are the same thing
  - A statement of the form $p(x_1, x_2, \ldots, x_n)$ is a proposition
- Examples
  - $p(x) \equiv x > 3$
    - p(4) has a truth value
  - q(y) ≡ y has paid his tuition
  - r(z) ≡ z has wireless access on campus
  - What does $r(x) \rightarrow q(x)$ mean?

Predicates, Continued

- Each propositional function takes a fixed number of arguments
  - older(x,y) ≡ x is older than y
  - Here "older" is being used as a propositional function

Formulas Involving Predicates

- Let $\phi(y) \equiv q(y) \rightarrow r(y)$
  - We say ϕ is *a formula in* y (not in text)
- Then $\phi(\text{Fred}) \equiv q(\text{Fred}) \rightarrow r(\text{Fred})$
  - This is an example of substitution of variables by values

Universe of Discourse and Quantifiers

- The *universe of discourse* or *domain* is the set of all possible values for variables
- We can refer to values in the universe either by using constant symbols (like "Fred") or by using quantifiers
- There are two quantifiers in standard predicate calculus: *for all* ($\forall$) and *there exists* ($\exists$)
- There are called the universal quantifier and the existential quantifier, respectively

Universal Quantifiers

- The universal quantification of p(x) is the following proposition:
  - "p(x) is true for all values of x in the universe of discourse"
  - Written $\forall x\, p(x)$ or $\forall x.p(x)$
- Similarly, if $\phi(x)$ is a formula in x, $\forall x.\phi(x)$ means the formula holds for all elements of the universe
  - What does $\forall x.(r(x) \rightarrow q(x))$ mean?
  - How is it different from $(\forall x.r(x)) \rightarrow (\forall x.q(x))$?