

Discrete Mathematical Structures CS 2233 Lecture Nine

Prof. William Winsborough
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Business

- Turn in:
 - Redone problem from Homework 2
 - Homework 3
- **Recall:** Homework 4, due Thursday 2/12
 - Section 2.1: 2a, 2b, 6, 8, 18
- Homework 5, due Tuesday 2/17
 - Will be accepted Thursday 2/19 as needed
 - Section 2.2: 2, 6, 10, 20, 26a
 - Section 2.3: 4, 10, 14, 16, 38
- Angela will run class and recitation Thursday 2/12
 - Winsborough is out of town
- Questions???

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Final Point About Proofs

- Constructive versus non-constructive proofs:
 - Consider problem 8 section 1.7:
 - Prove either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square
 - We show the following:
 - $\exists y. y \in \{2 \cdot 10^{500} + 15, 2 \cdot 10^{500} + 16\} \wedge \neg \exists z \in \mathbb{Z}. z^2 = y$
 - We do this by using $p \vee \neg p$:
 - p represents “ $2 \cdot 10^{500} + 15$ is a perfect square”
 - (Proof done on board)
 - This proof is **non-constructive**
 - We do not wind up with a witness
 - If we were to prove directly that $2 \cdot 10^{500} + 15$ is not a perfect square, this would be a constructive proof
 - We would wind up with a witness, namely $2 \cdot 10^{500} + 15$

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Sets

- $s \in S$ means s is an *element* of set S
- Given sets A and B ,
 - $A \subseteq B$ means A is a *subset* of B
 - This means $x \in A$ implies $x \in B$
 - $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
 - $A \subset B$ means A is a *proper subset* of B
 - $A \neq B$
 - $A = \{a_1, a_2, \dots\}$ means that A is enumerated by the sequence a_1, a_2, \dots
- *Set comprehension*: the set of values that satisfy a given property
 - Also called “set builder” notation
 - *E.g.*, $\text{EvenInts} = \{x \mid x \in \mathbb{Z} \wedge \exists z \in \mathbb{Z}. x = 2z\}$

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Basic Operators

- Empty set
 - $\emptyset = \{\}$
- Union
 - $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Intersection
 - $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Set difference
 - $A - B = \{x \mid x \in A \wedge \neg x \in B\}$
- Venn diagrams
- Universe U is the set consisting of all objects under consideration
 - Resembles a universe of discourse
- Compliment
 - $\bar{A} = U - A$
- Cartesian product:
 - $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

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Identities and Big Unions, Intersections

- Set Identities
 - Review Table 1, page 124
 - Review Example 11, page 125
- Given a collection of sets A_1, A_2, \dots, A_n

$$\bigcup_{1 \leq i \leq n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcap_{1 \leq i \leq n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

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Introduction to Functions

- Given sets A and B , a *function from A to B* is an assignment of exactly one element of B to each element of A
- For $a \in A$, we write $f(a) = b$ if $b \in B$ is the unique element associated with a by f
- We write $f : A \rightarrow B$ to indicate that f is a function from A to B
 - $A \rightarrow B$ is called the *type* of f

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Function Graphs

- This is a different usage of “graph” than either of these:
 - The plot of $f(x)$ on the Cartesian plane
 - A set of nodes and a set of edges
- Definition
 - Given a function $f : A \rightarrow B$, the *graph* of f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } b = f(a)\}$
- Given two sets, A and B , a *binary relation* r between A and B is a subset $r \subseteq A \times B$
- Formally speaking, a function **is** its graph
 - Thus, f is a subset of $A \times B$ (*i.e.*, a binary relation) in which each element of A occurs exactly once

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Formal Definition of Function

- A binary relation $r \subseteq A \times B$ is a *function* if
 - $\forall b \forall a_1 \forall a_2. ((a_1, b) \in r \wedge (a_2, b) \in r) \rightarrow a_1 = a_2$
 - In this case, we write $r(a)$ for b
- A function r is *total* (called a *total function*) if additionally
 - $\forall a \exists b. (a, b) \in r$

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Terminology

- Let $f : A \rightarrow B$
 - We say f *maps* A to B
 - A is the *domain* of f
 - B is the *codomain* of f
 - If $f(a) = b$,
 - b is the *image* of a and
 - a is the *pre-image* of b
 - If $S \subseteq A$, the *image* of S is $\{f(s) \mid s \in S\}$ and is denoted by $f(S)$
 - $f(A) = \{f(a) \mid a \in A\}$ is called the *range* of f

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Examples

- Successor function
 - $s : \mathbb{N} \rightarrow \mathbb{N}$
 - $s(n) = n+1$
- Floor
 - $\text{floor} : \mathbb{R} \rightarrow \mathbb{Z}$
 - $\text{floor}(x)$ = greatest integer less than x
- Square root
 - $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{C}$
 - The square root of a real is a complex number
- Truth assignment
 - $\sigma : \mathcal{P}^{\mathcal{A}} \rightarrow \{\text{true}, \text{false}\}$

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Injections and Surjections

- $f : A \rightarrow B$ is *one-to-one*, or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$
 - $f : A \xrightarrow{1:1} B$
 - $\forall a_1 \forall a_2 \forall b. ((a_1, b) \in f \wedge (a_2, b) \in f) \rightarrow a_1 = a_2$
 - Equivalently, $\forall a_1 \forall a_2. a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$
 - f is called an *injection*
- $f : A \rightarrow B$ is *onto*, or *surjective*, if and only if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$
 - f is a *function of A onto B*
 - $f : A \xrightarrow{\text{onto}} B$
 - $\forall b \exists a. (f(a) = b)$
 - f is called a *surjection*

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More Examples

- Injective and/or surjective?
 - Square
 - $f : Z \rightarrow Z$
 - $f(x) = x^2$
 - Double
 - $\text{twice} : Z \rightarrow Z$
 - $\text{twice}(x) = 2x$
 - Absolute value
 - $| \cdot | : Z \rightarrow N$
 - $|x| = \begin{cases} x & \text{if } 0 \leq x; \\ -x, & \text{otherwise} \end{cases}$

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Still More Examples

- Injective and/or surjective?
 - Integer successor
 - $zs : Z \rightarrow Z$
 - $zs(n) = n+1$
 - Mapping of 32-bit words to Z
 - Successor in $Z_3 = \{0, 1, 2\}$
 - $s(0) = 1$
 - $s(1) = 2$
 - $s(2) = 0$

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Bijections

- If $f : A \rightarrow B$ is injective and surjective, it is *bijective*
 - f is called a *one-to-one correspondence*, or a *bijection*

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Compositions

- Definition
 - Given $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of f and g , $f \circ g : A \rightarrow C$, is defined by $f \circ g(a) = f(g(a))$
 - Note that no special properties of f and g are required for $f \circ g$ to be defined.
 - For instance, f and g need not be injective, surjective, or bijective
 - However, if f and g have special properties, it often follows that $f \circ g$ special properties as well
 - Study hint: think through these relationships

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Function Inverses

- Theorem: Given $f : A \xrightarrow[1-1]{\text{onto}} B$, every $b \in B$ has a unique pre-image $a \in A$
- This justifies the following definition: Given

$$f : A \xrightarrow[1-1]{\text{onto}} B$$

- $f^{-1} : B \rightarrow A$ is the *inverse* of f
- $f^{-1}(b) = a$ iff $f(a) = b$

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Bijections and cardinality

- Sets A and B are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them
 - Notice that this defines a binary relation over sets
 - Contains the ordered pairs of sets between which there exists a one-to-one correspondence
- Pretty clear for finite sets

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