

Discrete Mathematical Structures CS 2233 Lecture Ten

Presented by Angela Dean

Prof. William Winsborough, Instructor

February 12, 2009

Business

- Turn in:
 - Homework 4
- **Recall:** Homework 5, due Tuesday 2/17
 - Will be accepted Thursday 2/19 as needed
 - Section 2.2: 2, 6, 10, 20, 26a
 - Section 2.3: 4, 10, 14, 16, 38
- **Recall:**
 - Review will be Tuesday 2/15
 - Do the mock exam by then
 - Solutions will be handed out in class
 - Bring your questions!!!
 - Midterm 1 will be Thursday 2/17
- Questions???

12 February 2009

Winsborough CS 2233 Lecture 10

2

Formal Definition of Function

- Given two sets, A and B, $r \subseteq A \times B$ is called a *binary relation*
 - Sometimes $(a,b) \in r$ is written $a r b$
 - Example: $< \subseteq \mathbb{R} \times \mathbb{R}$
 - $a < b$
- A binary relation $r \subseteq A \times B$ is a *function* if
 - $\forall a \forall b_1 \forall b_2. ((a,b_1) \in r \wedge (a,b_2) \in r) \rightarrow b_1 = b_2$
 - In this case, we write $r(a)$ for b
 - We will use names such as f, r , and sometimes S for functions
- A function r is *total* (called a *total function*) if additionally
 - $\forall a \exists b. (a,b) \in r$
- A function that is not total is called a *partial function*
 - Example: square root over the real numbers

12 February 2009

Winsborough CS 2233 Lecture 10

3

Terminology

- Let $f : A \rightarrow B$
 - We say f maps A to B or f is a *function from A to B*
 - A is the *domain* of f
 - B is the *codomain* of f
 - If $f(a) = b$,
 - b is the *image* of a
 - a is the *pre-image* of b
 - If $S \subseteq A$, the *image* of S is $\{f(s) \mid s \in S\}$ and is denoted by $f(S)$
 - $f(A) = \{f(a) \mid a \in A\}$ is called the *range* of f

12 February 2009

Winsborough CS 2233 Lecture 10

4

Examples

- Successor function
 - $s : \mathbb{N} \rightarrow \mathbb{N}$
 - $s(n) = n + 1$
- Floor
 - $\text{floor} : \mathbb{R} \rightarrow \mathbb{Z}$
 - $\text{floor}(x) =$ greatest integer less than x
- Square root
 - $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{C}$
 - The square root of a real is a complex number

12 February 2009

Winsborough CS 2233 Lecture 10

5

Injections and Surjections

- $f : A \rightarrow B$ is *one-to-one*, or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$
 - $f : A \rightarrow B$
 - $\forall a_1 \forall a_2 \forall b. ((a_1, b) \in f \wedge (a_2, b) \in f) \rightarrow a_1 = a_2$
 - Equivalently, $\forall a_1 \forall a_2. a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$
 - f is called an *injection*
- $f : A \rightarrow B$ is *onto*, or *surjective*, if and only if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$
 - f is a *function of A onto B*
 - $f : A \rightarrow B$
 - $\forall b \exists a. (f(a) = b)$ i.e., $f(A) = B$
 - f is called a *surjection*

12 February 2009

Winsborough CS 2233 Lecture 10

6

More Examples

- Injective and/or surjective?
 - Square
 - $f : \mathbb{Z} \rightarrow \mathbb{Z}$
 - $f(x) = x^2$
 - Double
 - $\text{twice} : \mathbb{Z} \rightarrow \mathbb{Z}$
 - $\text{twice}(x) = 2x$
 - Absolute value
 - $| \cdot | : \mathbb{Z} \rightarrow \mathbb{N}$
 - $|x| = \begin{cases} x & \text{if } 0 \leq x; \\ -x, & \text{otherwise} \end{cases}$

12 February 2009

Winsborough CS 2233 Lecture 10

7

Still More Examples

- Injective and/or surjective?
 - Integer successor
 - $zs : \mathbb{Z} \rightarrow \mathbb{Z}$
 - $zs(n) = n+1$
 - Mapping of 32-bit words to \mathbb{Z}
 - Successor in $\mathbb{Z}_3 = \{0, 1, 2\}$
 - $s(0) = 1$
 - $s(1) = 2$
 - $s(2) = 0$

12 February 2009

Winsborough CS 2233 Lecture 10

8

Bijections

- If $f : A \rightarrow B$ is injective and surjective, it is *bijective*
 - f is called a *one-to-one correspondence*, or a *bijection*

12 February 2009

Winsborough CS 2233 Lecture 10

9

Compositions

- Definition
 - Given $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of f and g , $f \circ g : A \rightarrow C$, is defined by $f \circ g(a) = f(g(a))$
 - Note that no special properties of f and g are required for $f \circ g$ to be defined.
 - For instance, f and g need not be injective, surjective, or bijective
 - However, if f and g have special properties, it often follows that $f \circ g$ special properties as well
 - Study hint: think through these relationships

12 February 2009

Winsborough CS 2233 Lecture 10

10

Function Inverses

- Theorem: Given $f : A \xrightarrow[\text{onto}]{1-1} B$, every $b \in B$ has a unique pre-image $a \in A$
- This justifies the following definition: Given

$$f : A \xrightarrow[\text{onto}]{1-1} B$$

- $f^{-1} : B \rightarrow A$ is the *inverse* of f
- $f^{-1}(b) = a$ iff $f(a) = b$

12 February 2009

Winsborough CS 2233 Lecture 10

11

Bijections and Cardinality

- Sets A and B are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them
 - Notice that this defines a binary relation over sets
 - Contains the ordered pairs of sets between which there exists a one-to-one correspondence
- Pretty clear for finite sets

12 February 2009

Winsborough CS 2233 Lecture 10

12