

Discrete Mathematical Structures CS 2233 Lecture Seven

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Business

- **Recall:** Homework 3 due Thursday 2/5:
 - Section 1.4: 2a, 2c, 4c-f, 10 (except h), 24a, 24d, 46
 - Work 4a, 4b, 10h together
 - In 4e, **explain your answers** and assume the usual interpretation for arithmetic and relational operators
 - Section 1.5: 4, 8, 16
 - Section 1.6: 2 (natural-language proof), 12, 18
 - Section 1.7: 4, 6, 8
- Problems on Homework 2
 - You may turn in Section 1.3 problem 44 again on Thursday 2/5
- Questions???

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Proof by Contradiction

- Prove a proposition p as follows:
 - Assume $\neg p$
 - Derive q and $\neg q$ for some proposition q
 - This is called a contradiction, since q and $\neg q$ cannot both be true
 - Conclude p
- See example 10, section 1.6, p80

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Rules of Inference for Universally Quantified Statements

- Universal instantiation

$$\frac{\forall x. p(x)}{p(c)}$$

- For any c

- Universal generalization

$$\frac{p(c)}{\forall x. p(x)}$$

- Must show $p(c)$ for arbitrary c

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Rules of Inference for Existentially Quantified Statements

- Existential instantiation

$$\frac{\exists x. p(x)}{p(c)}$$

- c must be a new name (constant) that does not appear earlier in the proof

- Existential generalization

$$\frac{p(c)}{\exists x. p(x)}$$

- c can be any name

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Universal Modus Ponens

- Combines propositional modus ponens with universal instantiation:

$$\frac{\forall x. \Phi(x) \rightarrow \Psi(x)}{\frac{\Phi(c)}{\Psi(c)}}$$

- Practice problems 14 and 15, p.73

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Proof Methods for Quantifiers

- Existence proofs (p91)
 - Constructive: find a witness
 - Nonconstructive: Use case analysis and the tautology $p \vee \neg p$
- Example 11, p91
- Practice Problem: 7, p102

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Proof by Case Analysis

- Suppose we want to prove $p \rightarrow q$
 - Further suppose that $p \equiv p_1 \vee p_2 \vee \dots \vee p_n$
- Note the following:
 - $p_1 \vee p_2 \vee \dots \vee p_n \rightarrow q \equiv$
 - $p_1 \rightarrow q \wedge p_2 \rightarrow q \wedge \dots \wedge p_n \rightarrow q$
- Putting these together, we see that to prove $p \rightarrow q$, it is sufficient to prove $p_1 \rightarrow q \wedge p_2 \rightarrow q \wedge \dots \wedge p_n \rightarrow q$
 - Special case:
 - When $p_1 \vee p_2 \vee \dots \vee p_n \equiv T$, this is a way of proving q

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Example Proof by Case Analysis

- Problem: give a natural-language proof of the following: $1+2+3+\dots+n = n(n+1)/2$ for all $n \in \mathbb{N}$
 - Proof structure: consider 2 cases
 - Case 1) n is even:

$$\begin{aligned} 1+2+3+\dots+n &= 1 + 2 + \dots + (n/2) + \\ &\quad n + (n-1) + \dots + ((n/2)+1) \\ &= \underbrace{(n+1) + (n+1) + \dots + (n+1)}_{n/2} \end{aligned}$$
 - Case 2) n is odd:

$$\begin{aligned} 1+2+3+\dots+n &= 0 + 1 + \dots + (n-1)/2 + \\ &\quad n + (n-1) + \dots + (n+1)/2 \\ &= \underbrace{n + n + \dots + n}_{(n+1)/2} \end{aligned}$$

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Sets

- $s \in S$ means s is an *element* of set S
- Given sets A and B ,
 - $A \subseteq B$ means A is a *subset* of B
 - This means $x \in A$ implies $x \in B$
 - $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
 - $A \subset B$ means A is a *proper subset* of B
 - $A \neq B$
 - $A = \{a_1, a_2, \dots\}$ means that A is enumerated by the sequence a_1, a_2, \dots
- *Set comprehension*: the set of values that satisfy a given property
 - E.g., $\text{EvenInts} = \{x \mid \exists z \in \mathbb{Z}. x = 2z\}$

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Basic Operators

- Empty set
 - $\emptyset = \{\}$
- Union
 - $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Intersection
 - $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Cartesian product:
 - $A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

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Identities and Notation: Big Unions and Intersections

- Set Identities
 - Review Table 1, page 124
 - Review Example 11, page 125
- Notation:
 - given a collection of sets A_1, A_2, \dots, A_n
 - $\bigcup_{1 \leq i \leq n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$
 - $\bigcap_{1 \leq i \leq n} A_i = A_1 \cap A_2 \cap \dots \cap A_n$

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