Discrete Mathematical Structures
CS 2233 Lecture Eight

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Business

- Turn in redone problem from Homework 2
- **Change in plans:** Homework 3 may be turned in either today or Tuesday 2/10
  - Today's lecture should help solving some problems
- Homework 4, due Thursday 2/12
  - Section 2.1: 2a, 2b, 6, 8, 18
- Read Sections 2.1, 2.2, 2.3
- Angela Dean (TA) Office Hours
  - Tuesday and Thursday 1-2pm SB 1.02.04
- Angela will run recitation today
  - Winsborough has a faculty meeting
- Angela will run class and recitation Thursday 2/12
  - Winsborough is out of town
- Questions???

Proof by Contradiction

- Prove a proposition p as follows:
  - Assume ¬p
  - Derive q and ¬q for some proposition q
    - This is called a contradiction, since q and ¬q cannot both be true
  - Conclude p
- See example 10, section 1.6, p80

Rules of Inference for Universally Quantified Statements

- Universal instantiation

$$\frac{\forall x.\ p(x)}{p(c)}$$

  - For any c
- Universal generalization

$$\frac{p(c)}{\forall x.\ p(x)}$$

  - Must show p(c) for arbitrary c

Rules of Inference for Existentially Quantified Statements

- Existential instantiation

$$\frac{\exists x.\ p(x)}{p(c)}$$

  - c must be a new name (constant) that does not appear earlier in the proof
- Existential generalization

$$\frac{p(c)}{\exists x.\ p(x)}$$

  - c can be any name

Universal Modus Ponens

- Combines propositional modus ponens with universal instantiation:

$$\frac{\begin{array}{l}\forall x.\Phi(x)\rightarrow\Psi(x)\\ \Phi(c)\end{array}}{\Psi(c)}$$

- Practice problems 14 and 15, p.73

Proof Methods for Quantifiers

- Existence proofs (p91)
  - Constructive: find a witness
  - Nonconstructive: Use case analysis and the tautology $p \lor \neg p$
- Example 11, p91
- Practice Problem: 7, p102

Proof by Case Analysis

- Suppose we want to prove $p \rightarrow q$
  - Further suppose that $p \equiv p_1 \lor p_2 \lor \ldots \lor p_n$
- Note the following:
  $p_1 \lor p_2 \lor \ldots \lor p_n \rightarrow q \equiv$
  $p_1 \rightarrow q \land p_2 \rightarrow q \land \ldots \land p_n \rightarrow q$
- Putting these together, we see that to prove $p \rightarrow q$, it is sufficient to prove
  $p_1 \rightarrow q \land p_2 \rightarrow q \land \ldots \land p_n \rightarrow q$
  - Special case:
    When $p_1 \lor p_2 \lor \ldots \lor p_n \equiv T$, this is a way of proving q

Example Proof by Case Analysis

- Problem: give a natural-language proof of the following: $1+2+3+\ldots+n = n(n+1)/2$ for all $n \in \mathbb{N}$
  - Proof structure: consider 2 cases
    - Case 1) n is even:
      $$1+2+3+\ldots+n = \begin{array}{l} 1 + 2 + \ldots + (n/2) + \\ n + (n-1) + \ldots + ((n/2)+1) \end{array} = \underbrace{(n+1) + (n+1) + \ldots + (n+1)}_{n/2}$$
    - Case 2) n is odd:
      $$1+2+3+\ldots+n = \begin{array}{l} 0 + 1 + \ldots + (n-1)/2 + \\ n + (n-1) + \ldots + (n+1)/2 \end{array} = \underbrace{n + n + \ldots + n}_{(n+1)/2}$$

Sets

- $s \in S$ means s is an *element* of set S
- Given sets A and B,
  - $A \subseteq B$ means A is a *subset* of B
    - This means $x \in A$ implies $x \in B$
  - $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
  - $A \subset B$ means A is a *proper subset* of B
    - $A \neq B$
  - $A = \{ a_1, a_2, \ldots \}$ means that A is enumerated by the sequence $a_1, a_2, \ldots$
- *Set comprehension:* the set of values that satisfy a given property
  - *E.g.,* EvenInts = $\{ x \mid \exists z \in Z. x=2z \}$

Basic Operators

- Empty set
  - $\varnothing = \{ \}$
- Union
  - $A \cup B = \{ x \mid x \in A \lor x \in B \}$
- Intersection
  - $A \cap B = \{ x \mid x \in A \land x \in B \}$
- Cartesian product:
  - $AxB = \{ (a,b) \mid a \in A \land b \in B \}$

Identities and Notation: Big Unions and Intersections

- Set Identities
  - Review Table 1, page 124
  - Review Example 11, page 125
- Notation:
  given a collection of sets $A_1, A_2, \cdots, A_n$

  $\bigcup_{1 \le i \le n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$

  $\bigcap_{1 \le i \le n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$