Discrete Mathematical Structures
CS 2233 Lecture Sixteen

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Business

- **Homework 7**
  - 4.1: 4, 6
  - 4.2: 4
- Read sections 4.1, 4.2 and 4.3
- Midterm II is in two weeks on 4/2

Mathematical Induction

- How can we show that a proposition P(n) holds for all natural numbers $n \in N$?
- Proof technique called *mathematical induction:*
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $P(k) \rightarrow P(k+1)$
- The proposition P(k) is called the *induction hypothesis*
- $(P(0) \wedge \forall k \in N.(P(k) \rightarrow P(k+1))) \rightarrow \forall n \in N.P(n)$

Example of Induction

- Theorem: $P(n) \equiv \sum_{0 \le i \le n} 2^i = 2^{n+1} - 1$
- Proof by induction
  - Basis: $P(0) \equiv \sum_{0 \le i \le 0} 2^i = 2^{0+1} - 1 \equiv 2^0 = 2-1$, which clearly holds
  - Step: We assume $P(k) \equiv \sum_{0 \le i \le k} 2^i = 2^{k+1} - 1$ and show $P(k+1) \equiv \sum_{0 \le i \le k+1} 2^i = 2^{k+2} - 1$ as follows:

$$\sum_{0 \le i \le k+1} 2^i = \sum_{0 \le i \le k} 2^i + 2^{k+1}$$
$$= (2^{k+1} - 1) + 2^{k+1} \text{ by the induction hypothesis}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

Validity of Induction

- Induction is valid because the natural numbers are well founded
  - Definition: A set is *well ordered* if each of its subsets has a least element
- Once basis and step are shown, the assumption that the property fails for some values yields a contradiction
  - Assume for contradiction that $P(0) \wedge \forall k(P(k) \rightarrow P(k+1)) \wedge \neg\forall n P(n)$
  - Consider the least $m \in N$ such that $\neg P(m)$
    - Case 1: if m = 0, the contradiction is immediate
    - Case 2: if m = k+1 for some $k \in N$, then by minimality of m P(k) holds. But this, together with the step, implies that $P(k+1) \equiv P(m)$ holds, giving us the desired contradiction

Strong Induction

- To show $\forall n \in N (P(n))$, the following is sufficient:
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $[P(0) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$
- This gives us a stronger induction hypothesis to use in the step
- It is valid for similar reasons to those shown on the previous slide

Example Proof by Strong Induction

- Match game:
  - Two players take turns removing any positive number of matches they wish from either of two piles
  - The player that removes the last match wins
- (Continued on next slide)

Example Proof by Strong Induction

- Assuming the two piles are initially the same size, the player that goes second can always win
  - Let P(n) be the proposition that the second player can win assuming n is the number of matches in both piles at the start
  - Basis: P(1). First player must remove one match from one pile, enabling the second player to win by removing the match from the other pile
  - Inductive step: $\forall k([P(0) \land \ldots \land P(k)] \rightarrow P(k+1))$. The second player removes the same number, r, of matches as did the first, but from the other pile. This leaves both piles having k+1-r matches. The induction hypothesis now guarantees the second player can win

Recursive Definition (Sec. 4.3)

- A *recursive definition* (also called an *inductive definition*) of a function f over N is given by specifying the value of the function in each of two cases:
  - The base case: f(0) is defined; more generally f(i) may be defined for all i less or equal to some $k \in N$
  - The recursive case: f(n+1) is defined in terms of f(n), f(n-1), …, f(0)
- Observe that f is a sequence

Examples

- Factorial F(n) = n!
  - F(0) = 1
  - F(n+1) = (n+1) F(n)
  - Defines the sequence {0!, 1!, 2!, …}
- Exponentiation
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$
- Σ: Sum of first n elements of a sequence $\{a_k\}$
  - $\sum_{0 \le i \le 0} a_i = 0$
  - $\sum_{0 \le i \le n+1} a_i = \sum_{0 \le i \le n} a_i + a_{n+1}$

Fibonacci Numbers

- The *Fibonacci numbers*, $f_0, f_1, f_2, \ldots$, are defined by:
  - $f_0 = 0$
  - $f_1 = 1$
  - $f_n = f_{n-1} + f_{n-2}$ for n > 1
- {0,1,1, 2, 3, 5, 8, …}

Inductive Proof about Recursively Defined Fibonacci Sequence

- Theorem: $\sum_{1 \le i \le n} f_{2i-1} = f_{2n}$, for n>0
- Basis (n=1): $\sum_{1 \le i \le 1} f_{2i-1} = f_1 = 0 + f_1 = f_0 + f_1 = f_2$
- Inductive Step (n+1):

$\sum_{1 \le i \le n+1} f_{2i-1} = \sum_{1 \le i \le n} f_{2i-1} + f_{2(n+1)-1}$

$= f_{2n} + f_{2n+1}$ by the induction hypothesis

$= f_{2n+2} = f_{2(n+1)}$