

Discrete Mathematical Structures CS 2233 Lecture Twenty-One

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Business

- Read 8.1, 8.4, 8.5
- **Recall:** Homework 10 due Thursday 4/23
 - 4.4: 8, 10, 22
 - Let $T(n)$ be the number of additions performed by the recursive algorithm for calculating $\text{fibonacci}(n)$, as presented in this lecture. Prove that $T(n) = O(2^n)$ by using strong induction to prove that $T(n) \leq 2^n$, for all $n \in \mathbb{N}$
 - 7.3: 2 (prove your answer is correct by using strong induction)

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Relations

- Let A_1, A_2, \dots, A_n be sets.
 - An n -ary relation on these sets is a subset of their Cartesian product: $r \subseteq A_1 \times A_2 \times \dots \times A_n$
- Binary (2-ary) relations are an important special case
 - Given sets A and B , r is a binary relation on A , and A_2 if $r \subseteq A_1 \times A_2$
 - For all $a_1 \in A_1$ and $a_2 \in A_2$, if $(a_1, a_2) \in r$, we write $a_1 r a_2$
- Representations of relations
 - List of tuples (pairs for binary)
 - n -dimensional matrix
 - Fast look up and delete of tuples
 - Not so good for sparse relations
 - A relation is sparse if its size is much smaller than that of the whole Cartesian product

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Relations as Functions and Sets

- As we have seen, functions are binary relations in which each value in A_1 occurs at most once in the first position of a tuple
 - In a *total function*, each value in A_1 occurs exactly once in the first position of some tuple
 - In a *partial function*, this is not the case
- Since relations are sets, all the set operations (e.g., \cup , \cap , $-$, \subset , \subseteq , $=$) can be applied to them
 - They can also be specified by using set builder notation (set comprehensions)

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Relations and Predicates

- In logic, predicates are interpreted in terms of relations
 - The interpretation of predicate p is relation r means:
 $p(a_1, a_2, \dots, a_n)$ is true
if and only if
 $(a_1, a_2, \dots, a_n) \in r$

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Properties of Binary Relations over a Single Set

- Let $r \subseteq A \times A$ be a binary relation over A
 - Reflexivity: r is *reflexive* if for all $a \in A$, $(a, a) \in r$ holds (i.e., is true)
 - Symmetry: a binary relation $r \subseteq A \times A$ is *symmetric* if for all $a_1, a_2 \in A$, $(a_1, a_2) \in r \rightarrow (a_2, a_1) \in r$
 - Transitivity: r is *transitive* if for all $a_1, a_2, a_3 \in A$, $(a_1, a_2) \in r \wedge (a_2, a_3) \in r \rightarrow (a_1, a_3) \in r$
- If r satisfies all three of these properties, it is called an *equivalence relation*

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Equivalence Classes and Partitions

- Let $r \subseteq A \times A$ be an equivalence relation over A
 - The equivalence class of any $a \in A$ is given by
 $[a] = \{a' \mid a' \in A \wedge (a, a') \in r\}$
 - So we have $[a] \subseteq A$
 - Theorem: for all $a_1, a_2 \in A$, either $[a_1]_r = [a_2]_r$ or $[a_1]_r \cap [a_2]_r = \emptyset$
- A set $\{A_1, A_2, \dots, A_n\}$ of subsets of A forms a partition if
 - For each i and j , $(A_i = A_j \vee A_i \cap A_j = \emptyset)$, and
 - $A_1 \cup A_2 \cup \dots \cup A_n = A$
- Theorem: if $r \subseteq A \times A$ is an equivalence relation, the set of equivalence classes of elements of A form defines a *partition* of A
 - The set of equivalence classes is $\{[a] \mid a \in A\}$