Discrete Mathematical Structures
CS 2233 Lecture Eighteen

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Business

- Read 4.3
- Homework 9
  - 4.3: 4, 6, 12, 22

Recursively Defined Sets

- Can define a set by giving some elements (basis) and giving a rule for constructing other elements from ones in the set (step)
  - Exclusion rule: such a construction defines the *smallest* set satisfying the basis and the step
- Example: Define S as follows
  - Basis: $3 \in S$
  - Step: if $x \in S$ and $y \in S$, then $x+y \in S$

Set of Strings

- Given an alphabet Σ the set of *strings* Σ^*
  - Basis: $\lambda \in \Sigma^*$ (λ denotes the empty string)
  - Step: if $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$ (concatenation)
- Examples:
  - Bit strings
  - words

Recursively Defined Sets

- Positive propositional formulas *Pos*
  - Basis: $p \in Pos$ for all propositional variables p
  - Recursive step: if $\Phi \in Pos$ and $\Psi \in Pos$, then $\Phi \wedge \Psi \in Pos$ and $\Phi \vee \Psi \in Pos$ and $(\Phi) \in Pos$
- Examples
  - $p \in Pos$
  - $p \wedge (q \vee r) \in Pos$
  - $\neg p \notin Pos$

Structural Induction

- Given a recursively defined set S, we can prove a property p(s) holds for all $s \in S$ by
  - Base case: proving that the property holds for elements added in the basis of the definition of S
  - Inductive step: proving that, if the property holds for the elements used to construct new elements in the recursive step, then the property holds for those new elements, too

Example Structural Induction

- Theorem: every formula $\Phi \in Pos$ is satisfied by σ_T, the truth assignment that makes all propositional variables true
- Proof:
  - Basis: the statement clearly holds for all propositional variables p
  - Step: if σ_T satisfies Φ and Ψ, then it also satisfies $\Phi \wedge \Psi$, $\Phi \vee \Psi$, and (Φ)

Recursively Defined Sets

- Structural induction is really just a shorthand for mathematical induction
  - Define a sequence of sets that converges to *Pos*
    - Pos_0 = the set of propositional variables
    - $Pos_{i+1} = Pos_i \cup \{ \Phi \wedge \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i\} \cup \{ \Phi \vee \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i\} \cup \{ (\Phi) \mid \Phi \in Pos_i\}$
  - $Pos = \bigcup_{i \in N} Pos_i$, which means $\Phi \in Pos$ if there exists a $k \in N$ such that $\Phi \in Pos_k$
- Now we can use mathematical induction to prove the theorem
  - Base case and inductive step correspond precisely to arguments used in structural induction

Another Fact about Positive Formulas

- Given truth assignments σ_1 and σ_2, we write $\sigma_1 \preccurlyeq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)
  - This is the point-wise "lifting" of the ordering that makes false < true
- Theorem: Consider any $\Phi \in Pos$, and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preccurlyeq \sigma_2$. If σ_1 satisfies Φ, then so does σ_2.
- Proof by structural induction (on next slide)

Proof

- Given $\Phi \in Pos$, and truth assignments such that $\sigma_1 \preccurlyeq \sigma_2$. If σ_1 satisfies Φ, then so does σ_2.
- Basis: $\Phi \in Pos$ is a propositional variable p. By def. of $\sigma_1 \preccurlyeq \sigma_2$, if σ_1 makes p true, so does σ_2
- Step: Assume that
  - if σ_1 satisfies Φ, then so does σ_2, and that
  - if σ_1 satisfies Ψ, then so does σ_2
  - It is easy to check the following, as required to complete the proof:
    - if σ_1 satisfies $\Phi \wedge \Psi$, then so does σ_2,
    - if σ_1 satisfies $\Phi \vee \Psi$, then so does σ_2, and
    - if σ_1 satisfies (Φ), then so does σ_2