Discrete Mathematical Structures
CS 3233 Lecture Seventeen

Prof. William Winsborough

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Business

- Assignment 6 due Thursday, October 20, 2pm, at Dr. Winsborough's office
  - Section 2.1: 2, 4, 12, 14, 16, 18, 20, 22, 26, 30, 36, 42

With Polynomials, the Degree is What Matters

- Theorem:
  If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, then $f(x)$ is $O(x^n)$

Example

- Sum of first n positive integers
  $1+2+\ldots+n \leq \underbrace{n+n+\ldots+n}_{n} = n^2$

  So $\sum_{i=1}^{n} i$ is $O(n^2)$
- It follows that the complexity of both bubble sort and insertion sort is $O(n^2)$

More Examples

- The factorial function, $f(n) = n!$, returns the product of the first n positive integers:
  $$n! = 1 \cdot 2 \cdot \ldots \cdot n \le \underbrace{n \cdot n \cdot \ldots \cdot n}_{n} = n^n$$
- So $n!$ is $O(n^n)$
  – Taking $C = 1$ and $k = 1$ as witnesses
- $\log n! \le \log n^n = n \log n$,
  – So $\log n!$ is $O(n \log n)$
  – Again, taking $C = 1$ and $k = 1$ as witnesses

Big-O and Sums

- Def: If f_1, f_2 are real-valued functions, (f_1+f_2) is the function such that that $(f_1+f_2)(x) = f_1(x)+f_2(x)$ for all x
- Theorem:
  If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1+f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|)$
- Corollary:
  If $f_1(x)$ and $f_2(x)$ are each $O(g(x))$, then $(f_1+f_2)(x)$ is $O(g(x))$

Big-O and Products

- Def: If f_1, f_2 are real-valued functions, (f_1f_2) is the function such that that $(f_1f_2)(x) = f_1(x) \cdot f_2(x)$ for all x
- Theorem:
  If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1f_2)(x)$ is $O(g_1(x)\, g_2(x))$

Example

- Give a Big-O estimate of $f(n) = 3n \log(n!) + (n^2 + 3) \log n$

Big-Omega and Big-Theta

- Big-O provides an upper bound on function growth
- Big-Omega gives a lower bound
  - Def: f(x) is $\Omega(g(x))$ if there are positive constants C and k such that $|f(x)| \geq C|g(x)|$ whenever $x>k$
- Big-Theta gives both
  - Def: f(x) is $\Theta(g(x))$ if f(x) is O(g(x)) and f(x) is $\Omega(g(x))$
  - In this case we say f(x) is *of order* g(x)

Example

- So $\sum_{i=1}^{n} i$ is $\Theta(n^2)$
  - We already shown it is $O(n^2)$, so we just have to show it is $\Omega(n^2)$
  - Summing only the terms greater than or equal to $\lceil n/2 \rceil$, we have $n - \lceil n/2 \rceil + 1$ such terms
  - So $1+2+\ldots+n \geq (n - \lceil n/2 \rceil + 1) \lceil n/2 \rceil$
    $\geq (n/2)(n/2) = n^2/4$