Discrete Mathematical Structures
CS 3233 Lecture 28

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Recursively Defined Sets

- Positive propositional formulas *Pos*
  - Basis: p $\in$ *Pos* for all propositional variables p
  - Recursive step: if $\Phi \in$ *Pos* and $\Psi \in$ *Pos*, then $\Phi \wedge \Psi \in$ *Pos* and $\Phi \vee \Psi \in$ *Pos* and $(\Phi) \in$ *Pos*
- Examples
  - p $\in$ *Pos*
  - $p \wedge (q \vee r) \in$ *Pos*
  - $\neg p \notin$ *Pos*

Structural Induction

- Given a recursively defined set S, we can prove a property p(s) holds for all $s \in S$ by
  - Base case: proving that the property holds for elements added in the basis of the definition of S
  - Inductive step: proving that, if the property holds for the elements used to construct new elements in the recursive step, then the property holds for those new elements, too

Example Structural Induction

- Theorem: every formula $\Phi \in \mathit{Pos}$ is satisfied by σ_T, the truth assignment that makes all propositional variables true
- Proof:
  - Basis: the statement clearly holds for all propositional variables p
  - Step: if σ_T satisfies Φ and Ψ, then it also satisfies $\Phi \wedge \Psi$, $\Phi \vee \Psi$, and (Φ)

Recursively Defined Sets

- Structural induction is really just a shorthand for mathematical induction
  - Define a sequence of sets that converges to *Pos*
    - Pos_0= the set of propositional variables
    - $Pos_{i+1} = Pos_i \cup \{ \Phi \wedge \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i\}$
      $\cup \{ \Phi \vee \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i\}$
      $\cup \{ (\Phi) \mid \Phi \in Pos_i\}$
  - $Pos = \bigcup_{i \in N} Pos_i$, which means $\Phi \in Pos$ if there exists a $k \in N$ such that $\Phi \in Pos_k$
- Now we can use mathematical induction to prove the theorem
  - Base case and inductive step correspond precisely to arguments used in structural induction

Another Fact about Positive Formulas

- Given truth assignments σ_1 and σ_2, we write $\sigma_1 \preccurlyeq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)
  - This is the point-wise extension of the ordering that makes true > false
- Theorem: Consider any $\Phi \in \mathit{Pos}$, and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preccurlyeq \sigma_2$. If σ_1 satisfies Φ, then so does σ_2.
- Proof by structural induction