Discrete Mathematical Structures
CS 3233 Lecture 32

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Business

- Assignment 12
  – Due 2pm Thursday 12/1 at Winsborough's Office
  – Section 5.1: 6, 8, 12
  – Section 5.2: 2, 6, 24, 26, 30
  – Section 5.3: 2, 6

Combinations

- Definition
  - Given a set S, an r-combination of S is an **unordered** arrangement of r distinct elements of S
- Examples
  - The number of sets of size three consisting of (distinct) alphabetic characters
  - The number of subsets of size r drawn from a set of size n
    - Compared to the set of one-to-one functions from a set of size r to a set of size n, we are considering only the range of the function, not its individual values

A Way of Thinking

- How many ways are there to order a set of size n?
- If you only care about the first r places in the ordering, (n-r)! of the orderings are effectively the same
  - This is because once I've chosen the first r places, there remain (n-r) elements whose order I don't care about
  - Thus, the number of permutations is n!/(n-r)!
  - For the number of combinations, you also don't care about the ordering of the elements in the first r places, so you divide the number of permutations by the number or ways of ordering the first r places, which is r!

Discrete Probability (Sec. 5.1)

- An *experiment* yields one of a given set of possible outcomes
- The *sample space* S is the set of possible outcomes
- An *event* E is a subset of the sample space
- Assuming each outcome in S is equally likely, the *probability* of event E is p(E) = |E|/|S|
  - This is just the sum of the probabilities of each outcome that belongs to the event

Simple Probability Example

- An urn contains 4 blue balls and 5 red balls. If one ball is chosen at random, what is the probability that it is blue?
- Rolling two dice, what is the probability of rolling a 7?
- Lottery: what is the probability of correctly choosing a set of 6 positive integers ≤ 40?
- What is the probability of being dealt 4 of a kind in a 5-card poker hand?

Combinations of Events

- Given an event $E \subseteq S$, $\bar{E}$ is the complementary event, S-E.
  $p(\bar{E}) = 1 - p(E)$
  – If a coin is flipped 10 times, what is the probability of getting heads at least once?
- If E_1 and E_2 are events in S then what is $p(E_1 \cup E_2)$?
  – What's the probability that a positive integer ≤ 100 and selected at random is divisible by neither 2 nor 5?

Probability Distributions

- What happens when all outcomes are not equally likely? Distribution is not *uniform*.
  - Given a finite sample space $S = \{x_1, \ldots, x_n\}$, a real-valued function $p: S \to [0,1]$ is a *probability distribution* if
    1. $0 \le p(x_i) \le 1$, for $i = 1,2,\ldots,n$
    2. $\sum_{0 \le i \le n} p(x_i) = 1$
- Biased-coin example: heads twice as likely as tails
- Probability of event E: $p(E) = \sum_{x \in E} p(x)$

Conditional Probability

- A fair coin is flipped 3 times. Assuming it comes up heads the first time, what is the probability it comes up tails an odd number of times?
- Definition
  - Let E and F be events with $p(F) > 0$. The conditional probability of E given F, denoted by $p(E|F)$, is $p(E|F) = p(E \cap F)/p(F)$
- Further examples
  - Suppose that a family has 2 children, one of which is a boy. What is the likelihood that the family has two boys?
  - Suppose that the first child is a boy. Now what is the likelihood that the family has two boys?

Independent Events

- A fair coin is again flipped three times. Does knowing it comes up tails the first time alter the probability that it comes up tails an odd number of times?
- Definition
  - Events E and F are independent if and only if $p(E \cap F) = p(E)p(F)$
- E = a family with 3 children has both sexes
  F = a family with 3 children has at most 1 boy
  Are E and F independent?

Random Variables

- A *random variable* X: S → R assigns a real number to each experiment outcome
- The distribution of a random variable X is the set of pairs (r, p(X=r)) for $r \in X(S)$, where p(X=r) is the probability that X takes on value r
- Let S be the 36 outcomes (i, j) obtained by rolling 2 dice. The customary value of a roll is the random variable i+j. What is the distribution of this random variable?

Algorithmic Example

- Suppose a linear search algorithm is used to find the position of a given value in a list of length n containing the positive integers $\leq n$ in arbitrary order
- The number of comparisons performed is a random variable
  - What is its distribution?
  - What is its expected value?
- In general, expected value of a random variable $X(s)$ is $E(X) = \sum_{s \in S} p(s)X(s)$