Discrete Mathematical Structures
CS 3233 Lecture 26

Prof. William Winsborough

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Business

- Assignment 9, due to Humayun in class on Wednesday 11/9
  - Section 2.4: 22
  - Section 3.3: 8, 14, 52
  - Use proof by contradiction and the well-foundedness of the natural numbers to show that strong induction is valid

Mathematical Induction

- How can we show that a proposition P(n) holds for all natural numbers $n \in N$?
- Proof technique called *mathematical induction:*
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $P(k) \rightarrow P(k+1)$
- The proposition P(k) is called the *induction hypothesis*
- $(P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)$

Example of Induction

- Theorem: $P(n) \equiv \sum_{0 \le i \le n} 2^i = 2^{n+1} - 1$
- Proof by induction
  - Basis: $P(0) \equiv \sum_{0 \le i \le 0} 2^i = 2^{0+1} - 1 \equiv 2^0 = 2\text{-}1$, which clearly holds
  - Step: We assume $P(k) \equiv \sum_{0 \le i \le k} 2^i = 2^{k+1} - 1$

    and show $P(k+1) \equiv \sum_{0 \le i \le k+1} 2^i = 2^{k+2} - 1$ as follows:

    $\sum_{0 \le i \le k+1} 2^i = \sum_{0 \le i \le k} 2^i + 2^{k+1}$

    $= (2^{k+1} - 1) + 2^{k+1}$ by the induction hypothesis

    $= 2 \cdot 2^{k+1} - 1$

    $= 2^{k+2} - 1$

Validity of Induction

- Induction is valid because the natural numbers are well founded
  - Definition: A set is *well founded* if each of its subsets has a least element
- Once basis and step are shown, the assumption that the property fails for some values yields a contradiction
  - Assume for contradiction that $P(0) \land \forall k(P(k) \rightarrow P(k+1)) \land \neg\forall n P(n)$
  - Consider the least $m \in N$ such that $\neg P(m)$
    - Case 1: if $m = 0$, the contradiction is immediate
    - Case 2: if $m = k+1$ for some $k \in N$, then by minimality of m P(k) holds. But this, together with the step, implies that $P(k+1) \equiv P(m)$ holds, giving us the desired contradiction

Strong Induction

- To show $\forall n \in N\ (P(n))$, the following is sufficient:
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $[P(0) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$
- This gives us a stronger induction hypothesis to use in the step
- It is valid for similar reasons to those shown on the previous slide