

# Discrete Mathematical Structures

## CS 3233 Lecture Twelve

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# Function Inverses

- Theorem: Given  $f : A \xrightarrow[\text{onto}]{1-1} B$ , every  $b \in B$  has a unique pre-image  $a \in A$
- This justifies the following definition: Given

$$f : A \xrightarrow[\text{onto}]{1-1} B$$

–  $f^{-1} : B \rightarrow A$  is the *inverse of*

–  $f^{-1}(b) = a$  iff  $f(a) = b$

# Compositions

- Definition
  - Given  $g : A \rightarrow B$  and  $f : B \rightarrow C$ , the composition of  $f$  and  $g$ ,  $f \circ g : A \rightarrow C$ , is defined by
$$f \circ g(a) = f(g(a))$$
  - Note that no special properties of  $f$  and  $g$  are required for  $f \circ g$  to be defined.
    - For instance,  $f$  and  $g$  need not be injective, surjective, or bijective
    - However, if  $f$  and  $g$  have special properties, it often follows that  $f \circ g$  special properties as well
    - Study hint: think through these relationships

# Review: Bijections and Cardinality

- Sets  $A$  and  $B$  are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them

# Countability

- Definition
  - A set  $A$  is *countable* if it is finite or it is equinumerous to  $\mathbb{Z}^+$  (or  $\mathbb{N}$ )
  - Otherwise,  $A$  is *uncountable*

# Sequences

- Definition

- A *sequence* is a function from  $\mathbb{N}$  (or  $\mathbb{Z}^+$ ) to a given set  $S$

- We use  $a_n$  to denote the image of  $n$

- We use  $\{a_n\}$  to denote the whole sequence

- Less formally, we sometimes denote it by  $\{a_0, a_1, a_2, a_3, \dots\}$

- If the function is onto we say  $\{a_n\}$  enumerates  $S$

# Countability and Enumeration

- Theorem:  $S$  is countable if and only if there exists a sequence that enumerates  $S$
- Proof
  - only if: If there is a bijection between  $\mathbb{N}$  and  $S$ , it is a sequence that enumerates  $S$
  - if: Given a sequence that enumerates  $S$ , either  $S$  is finite or dropping repeated values from the sequence yields a bijection between  $\mathbb{N}$  and  $S$

# Exercise 33, Section 3.2

- Theorem: If  $A$  is an uncountable set and  $B$  is a countable set,  $A - B$  is uncountable
- Proof
  - Suppose for contradiction that  $A - B$  is countable
  - This means that there is a sequence that enumerates all elements of  $A - B$
  - We can now construct a sequence that enumerates  $A$ 
    - It alternates between the sequence that enumerates  $A - B$  and the sequence that enumerates  $B$
    - This contradicts the assumption that  $A$  is uncountable
  - It follows that the assumption  $A - B$  is countable is false, hence  $A - B$  is uncountable

# Exercise 35, Section 3.2

- Theorem: If  $A$  is an uncountable set and  $A$  is a subset of  $B$ , then  $B$  is uncountable
- Proof:
  - Suppose  $B$  is countable
  - This means there is a sequence that enumerates  $B$
  - A sequence that enumerates  $A$  can be constructed by dropping the elements of  $B - A$ , yielding the desired contradiction

# Uncountability of the Reals

- Theorem
  - $\mathbb{R}$ , the set of real numbers, is uncountable
- Proof
  - Uses Georg Cantor's *diagonalization argument*
  - Outline
    - Assume for contradiction that the real interval  $[0,1]$  is enumerated by  $\{a_n\}$
    - Use  $\{a_n\}$  to construct a real in  $[0,1]$  that does not occur in  $\{a_n\}$
    - Idea: for each decimal place,  $n$ , in the representation of the constructed value, choose a decimal different from the  $n^{\text{th}}$  place of  $a_n$
    - The fact that the constructed value differs from each value  $a_n$  shows that  $\{a_n\}$  does not enumerate  $[0,1]$
    - Now use Exercise 35 to complete the proof that  $\mathbb{R}$  is uncountable