Solutions: Mock Exam for Final
Discrete Mathematical Structures CS3233

December, 2005

The final exam is cumulative. This mock exam covers material starting with that presented in lecture 11/07/2005, which was not previously tested. This includes material in sections 3.4, 3.5, 4.1-3, and 5.1-3. Students should refer to midterms 1 and 2, and to the corresonding mock exams and homeworks, for assistance in studying material previously tested.

1. Determine whether the following are valid recursive definitions of a function $f : N \to Z$:

   (a) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n-2)$ for $n > 1$

   (b) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n)$ for $n > 1$

   (c) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n+1) + f(n+2)$ for $n > 1$

   Solution: valid, invalid, invalid

2. Let the set of positive propositional formulas Pos be recursively defined as follows:
   Basis: $p \in$ *Pos* for all propositional variables p.
   Recursive step: if $\varphi \in$ *Pos* and $\psi \in$ *Pos*, then $\varphi \wedge \psi \in$ *Pos*, $\varphi \vee \psi \in$ *Pos*, and $(\varphi) \in$ *Pos*.

   Given truth assignments σ_1 and σ_2, we write $\sigma_1 \preceq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)

   Prove the following by structural induction: Consider any $\varphi \in$ *Pos* and and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preceq \sigma_2$. If σ_1 satisfies φ, then so does σ_2.

   Solution:
   Basis: If φ is a propositional variable and σ_1 satisfies φ, then so does σ_2 by definition of $\sigma_1 \preceq \sigma_2$.

   inductive step: Suppose for the induction assumption that if σ_1 satisfies φ then σ_2 satisfies φ and that if σ_1 satisfies ψ then σ_2 satisfies ψ. We show that if σ_1 satisfies $\varphi \vee \psi$ then σ_2 satisfies $\varphi \vee \psi$. A complete answer to this problem also requires one to prove that if σ_1 satisfies $\varphi \vee \psi$ then σ_2 satisfies $\varphi \vee \psi$ and that if σ_1 satisfies (φ) then σ_2 satisfies (φ). We assume the atecedent and show the consequent. Since σ_1 satisfies $\varphi \vee \psi$, it must be that σ_1 satisfies φ or σ_1 satisfies ψ. In the former case it follows by using the induction assumption that σ_2 satisfies φ, from which it follows that σ_2 satisfies $\varphi \vee \psi$, as desired. We obtain the desired conclusion similarly in the case that σ_1 satisfies ψ.

3. What is the best big-O estimate of the worst-case complexity of mergesort? Justify your answer.

   Solution: $\mathcal{O}(n \log n)$. Refer to lecture notes from 11/14 for a more detailed explanation. The main point is that unless the input list is of length 1, each invocation of mergesort splits the input list in half and feeds each half to a recursive call. A list of length n can be split in half at most $\log n$ times, leading to $\mathcal{O}(\log n)$ levels in the recursion. After the recursive calls return, $\mathcal{O}(n)$ comparisons are performed to merge the lists at each level of recursion.

4. In this section of the exam, please do not fully evaluate your expressions. Simplify your answers so that they are expressed in terms of factorials and exponents, but not in terms of $C(n,r)$ or $P(n,r)$.

   (a) How many r-letter words can be constructed using an alphabet of size n?
   Solution: n^r

   (b) How many ways are there to order a set of size n?
   Solution: $n!$

(c) How many ways are there to order a set of size n if the first r places are always occupied by the same r elements?

Solution: $(n-r)!$

(d) How many ways are there to partition a set of size n into two sets, one of which has size r? (A *partition* of set S is a collection of subsets of S such that the intersection of any two sets in the collection is empty and the union of all the sets in the collection is the whole original set.)

Solution: $C(n, r) = \frac{n!}{r!(n-r)!}$.

(e) How many ways are there to partition a set of size n into two sets A and B? Note that it matters which set is called A and which one is called B.

Solution: One way of thinking about this problem is to use solution to the previous problem and to sum over the different sizes of A:

$$\sum_{0 \leq i \leq n} C(n, i) = \sum_{0 \leq i \leq n} \frac{n!}{i!(n-i)!}$$

However, a more direct analysis just counts the number of subsets of n, which is 2^n.

Either answer would get full points and both are correct. This reflects an important identity that we did not have time to discuss in lecture, namely

$$\sum_{0 \leq i \leq n} \frac{n!}{i!(n-i)!} = 2^n$$

(f) How many 8-bit sequences have exactly 3 0's?

Solution: We just need to count the number of ways of selecting the three positions occupied by 0's: $C(8, 3) = \frac{8!}{3!5!}$.

(g) How many ways are there to put 6 keys on a circular key ring? The two sides of the ring are the same and the two sides of the keys are the same. (So both ring and keys are symetrical.) There is also no way to tell which key is first.

Solution: Since the two sides of the keys are the same, we don't have to consider which way they are turned. There are 6! ways of linearly ordering the keys. However, each arrangement of the 6 keys around the ring generates 6 different linear orderings, depending on which key you consider to be first. So, if sides of the key ring could be distinguished, there would be $\frac{6!}{6} = 5!$ ways of arranging the keys on the ring. But since each arrangement is considered the same as the (otherwise different) one obtained from it by turning over the loaded ring, we have to divide this number of 2, obtaining the answer $\frac{5!}{2}$.

(h) What is the probability of being dealt a particular hand of 5 cards from a deck of 52 distinct cards?

Solution: $\frac{1}{C(52,5)}$

(i) What is the conditional probability of being dealt a particular hand of 5 cards from a deck of 52 distinct cards given that another player is also dealt a hand of 5 cards?

Solution: Since the hand dealt to the other player can be any hand, the fact that 5 cards of the 47 cards not dealt to you are dealt to someone else has no effect on the probability of your receiving any given card. So the the conditional probability is the same as the unconditional probability of being dealt a certain hand: $\frac{1}{C(52,5)}$

(j) Suppose a permutation of all 26 lowercase letters is selected at random with each permutation equally likely.

i. What is the probability that the first 13 letters of the permutation are in order?
Solution: Whatever the first 13 letters may be, there are $13!$ ways of ordering them, only one of which is alphabetically ordered. Thus the probability is 1/13! A more brute force way of getting the answer is to count the number of permutations in which the first 13 letters are in order and to divide this by the total number of permutations. There are C(26,13) ways of choosing 13 letters to put in the first 13 positions. For each of these choices, only one ordering of the selected letters is alphabetical. The 13 letters not chosen for the first 13 positions can be placed in 13! different orders. So the probability is $\frac{C(26,13)13!}{26!} = \frac{1}{13!}$

ii. What is the probability that a and b are not next to each other?
Solution: There are $24!$ ways of ordering the letters c through z. We must choose two places to insert the letters a and b out of 25 places either between two of the other 24 letters or before or after all of them. (By choosing 2 places, we ensure a and b are not ajacent.) We must also choose which of the two places gets a and which gets b. So the number of permutations that do not have a and b ajacent is $2(24!C(25,2))$. Dividing this by 26! yields $\frac{12}{13}$.

(k) Given a 6-sided die for which each of the values 1 through 6 comes up with equal probability, what is the conditional probability of rolling a 6 given that the last 3 times you rolled you also got a 6.
Solution: $\frac{1}{6}$

(l) You and Humayun are playing poker, a game which he claims not to understand. You are each simultaneously and fairly dealt hands of 5 cards from the same 52-card deck. There are only 4 ways of getting a royal flush and these 4 hands do not share any cards. What is the conditional probability of your being dealt a royal flush given that Humayun is also dealt a royal flush?
Solution:

$$\frac{\frac{C(4,2)}{C(52,5)C(47,5)}}{\frac{C(4,1)}{C(52,5)}} = \frac{C(4,2)}{4C(47,5)}$$

(m) What is the expected value of a roll of one 6-sided die assuming each of the values 1 through 6 comes up with equal probability?
Solution: 3.5

(n) Given a 6-sided die that is weighted so that each of the values 2 through 6 comes up with equal probability, but 1 comes up twice as often as the other values, what is the probability distribution for a roll of this die? (Give the probability of getting each one of the 6 values.)
Solution: $(1, \frac{2}{7}), (2, \frac{1}{7}), (3, \frac{1}{7}), (4, \frac{1}{7}), (5, \frac{1}{7}), (6, \frac{1}{7})$

(o) What is the expected value of a roll of one 6-sided die weighted as described in the previous problem?
Solution: $\frac{22}{7}$