Discrete Mathematical Structures

CS 3233 Lecture 17

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Business

- Messages from Winsborough
  - I'm very sorry I was unable to meet you last Thursday. I'm having some health problems
    - Students are permitted to turn in #6, sec. 4.1 and #4, sec. 4.2 on Thursday
  - I'm worried that people didn't do very well on homework 6
    - Even if you are in the Thursday recitation, I encourage you to attend today's recitation, especially if you are still having trouble with big-O
- Homework 8 due Tuesday 7 November
  - 4.3: 4, 6, 12
- Read section 4.3. Review in section 4.1 examples 8 and 9
- Mock exam
  - Posted on the course web site
  - Solutions will be posted around the end of the week (because you are supposed to work the exam without the solutions)

Mathematical Induction

- How can we show that a proposition P(n) holds for all natural numbers $n \in N$?
- Proof technique called *mathematical induction:*
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $P(k) \rightarrow P(k+1)$
- The proposition P(k) is called the *induction hypothesis*
- The proof rule is based on the following tautology:
  $(P(0) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)$

Example of Induction

- Theorem: Given natural number n and $r \neq 1$,
  $P(n) \equiv \sum_{0 \le i \le n} ar^i = (ar^{n+1} - a)/(r-1)$
- Proof by induction
  - Basis: $P(0) \equiv \sum_{0 \le i \le 0} ar^i = (ar^{0+1} - a)/(r-1)$,
    which holds because $(ar^1 - a)/(r-1) = a(r-1)/(r-1) = a$
  - Step: Using the induction hypothesis $P(k) \equiv \sum_{0 \le i \le k} ar^i = (ar^{k+1} - a)/(r-1)$
    we show $P(k+1) \equiv \sum_{0 \le i \le k+1} ar^i = (ar^{k+2} - a)/(r-1)$ as follows:

$$\sum_{0 \le i \le k+1} ar^i = \sum_{0 \le i \le k} ar^i + ar^{k+1}$$

$= (ar^{k+1} - a)/(r-1) + ar^{k+1}$ by the induction hypothesis

$$= (ar^{k+1} - a)/(r-1) + ar^{k+1} (r-1)/(r-1)$$

$$= (ar^{k+1} - a)/(r-1) + (ar^{k+2} - ar^{k+1})/(r-1)$$

$$= (ar^{k+2} - a)/(r-1)$$

Validity of Mathematical Induction

- Induction is valid because the natural numbers are well founded
  - Definition: A set is *well ordered* if each of its subsets has a least element
- Induction streamlines a proof by contradiction
  - Suppose the basis and step hold: $P(0) \land \forall k(P(k) \to P(k+1))$
  - The assumption that the property fails to hold everywhere $\neg\forall n P(n)$ enables us to prove a contradiction:
  - Consider the least $m \in N$ such that $\neg P(m)$
    - Case 1: if $m = 0$, the contradiction is immediate
    - Case 2: if $m = k+1$ for some $k \in N$, then by minimality of m $P(k)$ holds. But this, together with the step, implies that $P(k+1) \equiv P(m)$ holds, giving us the desired contradiction

Strong Induction (Sec. 4.2)

- To show $\forall n \in N\ (P(n))$, the following is sufficient:
  - Basis: show P(0)
  - Inductive Step: show that for all $k \in N$, $[P(0) \land \ldots \land P(k)] \to P(k+1)$
- This gives us a stronger induction hypothesis to use in the step
- Note that every proof by mathematical induction is also a proof by strong induction
- Although we won't prove it here, everything that can be proven by using strong induction can also be proven by using mathematical induction
  - But it's often not as convenient

Example Proof by Strong Induction

- Match game:
  - Two players take turns removing any positive number of matches they wish from either of two piles
  - The player that removes the last match wins

Example Proof by Strong Induction

- Assuming the two piles are initially the same size, the player that goes second can always win
  - Let P(n) be the proposition that the second player can win assuming n is the number of matches in both piles at the start
  - Basis: P(1). First player must remove one match from one pile, enabling the second player to win by removing the match from the other pile
  - Inductive step: $\forall k([P(0) \land \ldots \land P(k)] \to P(k+1))$. The second player removes the same number, r, of matches as did the first, but from the other pile. This leaves both piles having k+1-r matches. The induction hypothesis now guarantees the second player can win

Recursive Definition (Sec. 4.3)

- A *recursive definition* (also called an *inductive definition*) of a function f over N is given by specifying the value of the function in each of two cases:
  - The base case: f(0) is defined; more generally f(i) may be defined for all i less or equal to some $k \in N$
  - The recursive case: f(n+1) is defined in terms of f(n), f(n-1), …, f(0)
- Observe that f is a sequence

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Examples

- Factorial F(n) = n!
  - F(0) = 1
  - F(n+1) = (n+1) F(n)
  - Defines the sequence {0!, 1!, 2!, …}
- Exponentiation
  - $a^0 = 1$
  - $a^{n+1} = a \cdot a^n$
- Σ: Sum of first n elements of a sequence $\{a_k\}$
  - $\sum_{0 \le i \le 0} a_i = 0$
  - $\sum_{0 \le i \le n+1} a_i = \sum_{0 \le i \le n} a_i + a_{n+1}$

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Fibonacci Numbers

- The *Fibonacci numbers*, f_0, f_1, f_2, …, are defined by:
  - $f_0 = 0$
  - $f_1 = 1$
  - $f_n = f_{n-1} + f_{n-2}$ for n > 1
- {1, 2, 3, 5, 8, …}

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Inductive Proof about Recursively Defined Fibonacci Sequence

- Theorem: $\sum_{1 \le i \le n} f_{2i-1} = f_{2n}$, for n>0
- Basis (n=1): $\sum_{1 \le i \le 1} f_{2i-1} = f_1 = 0 + f_1 = f_0 + f_1 = f_2$
- Inductive Step (n+1):

  $\sum_{1 \le i \le n+1} f_{2i-1} = \sum_{1 \le i \le n} f_{2i-1} + f_{2(n+1)-1}$

  $= f_{2n} + f_{2n+1}$ by the induction hypothesis

  $= f_{2n+2} = f_{2(n+1)}$

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