

Discrete Mathematical Structures CS 3233 Lecture 18

Prof. William Winsborough
November 2, 2006
Delivered by: Prof. Thomas Bylander

Business

- Recall: Homework 8 due Tuesday 7 Nov.
 - 4.3: 4, 6, 12
- Recall: Exam in class Thursday 9 Nov.
- Review session Tuesday 7 Nov.
 - Prepare by figuring out what to ask Winsborough to review
 - Use: mock exam, quizzes, homeworks, lecture notes
- Mock exam
 - Solutions will be posted around the end of the week, but begin by working the exam without using the solutions

2 November 2006

Winsborough CS 3233 Lecture 18

2

Recursively Defined Sets

- Can define a set by giving some elements (basis) and giving a rule for constructing other elements from ones in the set (step)
 - Exclusion rule: such a construction defines the *smallest* set satisfying the basis and the step
- Example: Define S as follows
 - Basis: $3 \in S$
 - Step: if $x \in S$ and $y \in S$, then $x+y \in S$

2 November 2006

Winsborough CS 3233 Lecture 18

3

Set of Strings

- Given an alphabet Σ the set of *strings* Σ^*
 - Basis: $\lambda \in \Sigma^*$ (λ denotes the empty string)
 - Step: if $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$ (concatenation)
- Examples:
 - Bit strings
 - words

2 November 2006

Winsborough CS 3233 Lecture 18

4

Recursively Defined Sets

- Positive propositional formulas Pos
 - Basis: $p \in Pos$ for all propositional variables p
 - Recursive step: if $\Phi \in Pos$ and $\Psi \in Pos$, then $\Phi \wedge \Psi \in Pos$ and $\Phi \vee \Psi \in Pos$ and $(\Phi) \in Pos$
- Examples
 - $p \in Pos$
 - $p \wedge (q \vee r) \in Pos$
 - $\neg p \notin Pos$

2 November 2006

Winsborough CS 3233 Lecture 18

5

Structural Induction

- Given a recursively defined set S , we can prove a property $p(s)$ holds for all $s \in S$ by
 - Base case: proving that the property holds for elements added in the basis of the definition of S
 - Inductive step: proving that, if the property holds for the elements used to construct new elements in the recursive step, then the property holds for those new elements, too

2 November 2006

Winsborough CS 3233 Lecture 18

6

Example Structural Induction

- Theorem: every formula $\Phi \in Pos$ is satisfied by σ_T , the truth assignment that makes all propositional variables true
- Proof:
 - Basis: the statement clearly holds for all propositional variables p
 - Step: if σ_T satisfies Φ and Ψ , then it also satisfies $\Phi \wedge \Psi$, $\Phi \vee \Psi$, and (Φ)

2 November 2006

Winsborough CS 3233 Lecture 18

7

Recursively Defined Sets

- Structural induction is really just a shorthand for mathematical induction
 - Define a sequence of sets that converges to Pos
 - $Pos_0 =$ the set of propositional variables
 - $Pos_{i+1} = Pos_i \cup \{ \Phi \wedge \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i \}$
 $\cup \{ \Phi \vee \Psi \mid \Phi \in Pos_i \text{ and } \Psi \in Pos_i \}$
 $\cup \{ (\Phi) \mid \Phi \in Pos_i \}$
 - $Pos = \bigcup_{i \in \mathbb{N}} Pos_i$, which means $\Phi \in Pos$ if there exists a $k \in \mathbb{N}$ such that $\Phi \in Pos_k$
- Now we can use mathematical induction to prove the theorem
 - Base case and inductive step correspond precisely to arguments used in structural induction

2 November 2006

Winsborough CS 3233 Lecture 18

8

Another Fact about Positive Formulas

- Given truth assignments σ_1 and σ_2 , we write $\sigma_1 \preceq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)
 - This is the point-wise “lifting” of the ordering that makes false < true
- Theorem: Consider any $\Phi \in Pos$, and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preceq \sigma_2$. If σ_1 satisfies Φ , then so does σ_2 .
- Proof by structural induction (on next slide)

2 November 2006

Winsborough CS 3233 Lecture 18

9

Proof

- Given $\Phi \in Pos$, and truth assignments such that $\sigma_1 \preceq \sigma_2$. If σ_1 satisfies Φ , then so does σ_2 .
- Basis: $\Phi \in Pos$ is a propositional variable p . By def. of $\sigma_1 \preceq \sigma_2$, if σ_1 makes p true, so does σ_2
- Step: Assume that
 - if σ_1 satisfies Φ , then so does σ_2 , and that
 - if σ_1 satisfies Ψ , then so does σ_2
 - It is easy to check the following, as required to complete the proof:
 - if σ_1 satisfies $\Phi \wedge \Psi$, then so does σ_2 ,
 - if σ_1 satisfies $\Phi \vee \Psi$, then so does σ_2 , and
 - if σ_1 satisfies (Φ) , then so does σ_2

2 November 2006

Winsborough CS 3233 Lecture 18

10