Discrete Mathematical Structures
CS 3233 Lecture One

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Course Introduction

- See syllabus:
  - http://www.cs.utsa.edu/~winsboro/teaching/CS3233F2006/Syllabus.htm
- Attendance is mandatory in lecture and recitation
- Read the text book
  - Lectures are associated with sections in the text as indicated on the syllabus
- If you cannot make my office hours, please set up an appointment
  - Email, call, drop by
- Please speak up!

Section 1.1: Introduction to Logic

- Aim: formalize mathematical argument
- *Propositional logic* (also called *propositional calculus*)
  - Deals with propositions
  - A proposition is a declarative sentence that is either *true* or *false*, but not both

Propositions

- Some sentences that are propositions
  - 1+1 = 2
  - Today is Saturday
  - Will is a new professor and San Antonio is the center of the universe
- Some sentences that are not propositions
  - Is this a CS class?
    - Not declarative
  - Read your syllabus carefully
    - Not declarative
  - $x + 5 = 7$
    - Neither true or false, since the truth value depends on the value assigned to x

Negation

- Def: Given a proposition p, ¬p denotes the negation of p
  - ¬p means “it is not the case that p”
- Truth table for ¬p:

p	¬p
T	F
F	T

Given truth assignments for p

Resulting truth value of ¬p

Conjunction

- Def: Given propositions p and q, $p \wedge q$ denotes the conjunction of p and q
  - $p \wedge q$ means "p and q"
- Truth table for $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Given truth assignments

Resulting truth value

Disjunction

- Def: Given propositions p and q, p ∨ q denotes the disjunction of p and q
  - p ∨ q means "p or q " (*inclusive or*)
- Truth table for p ∨ q:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Given truth assignments

Resulting truth value

Implication

- Def: Given propositions p and q, $p \rightarrow q$ is an implication
  – $p \rightarrow q$ means "p implies q "
  – p is the *hypothesis, antecedent* or *premise*
  – q is the *conclusion, consequence,* or *consequent*
  – Truth table for $p \rightarrow q$:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Understanding Implications

- Some readings of $p \rightarrow q$:
  - if p, then q
  - q if p
  - q when p
  - p only if q
  - q follows from p
  - q is a necessary condition for p
  - p is a sufficient condition for q
  - q is necessary for p
  - p is sufficient for q

Implications Related to $p \rightarrow q$

- *Contrapositive*: $\neg q \rightarrow \neg p$
- *Converse*: $q \rightarrow p$
- *Inverse*: $\neg p \rightarrow \neg q$

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Exclusive Or

- Def: Given propositions p and q, p ⊕ q denotes the exclusive or of p and q
  - p ⊕ q means "p or q, but not both"
- Truth table for p ⊕ q:

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Biconditionals

- Def: Given propositions p and q, $p \leftrightarrow q$ is a *biconditional*
  - $p \leftrightarrow q$ means "p if and only if q"
- Truth table for $p \leftrightarrow q$:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Translating English to Logic

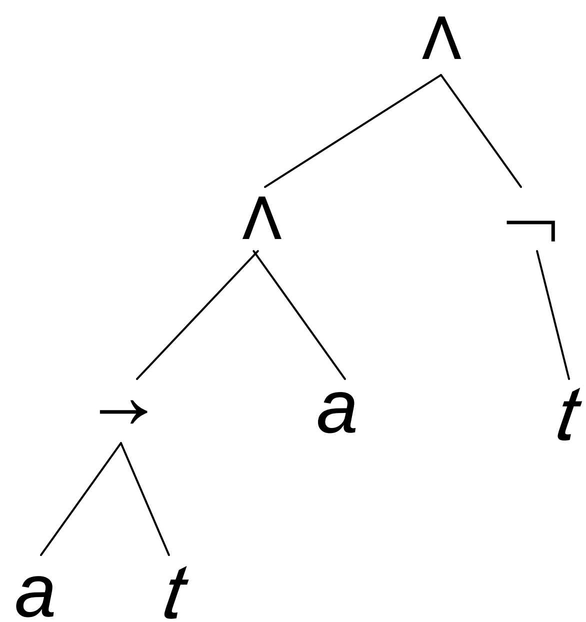
- Fred can access the wireless network only if Fred has paid his tuition
  - Let *a* represent "Fred can access wireless"
  - Let *t* represent "Fred has paid his tuition"
  - $a \rightarrow t$

Syntax of Propositional Formulas

- Definition of *propositional formula:*
  - A propositional variable *p* is a propositional formula
  - The constants **T** and **F** are propositional formulas
  - If ϕ and ψ are propositional formulas, then the following are also propositional formulas:
    - (ϕ)
    - $\neg\phi$
    - $\phi \wedge \psi$
    - $\phi \vee \psi$
    - $\phi \rightarrow \psi$
    - $\phi \leftrightarrow \psi$
    - $\phi \oplus \psi$

Expression Trees

- Example: $((a \rightarrow t) \wedge a) \wedge \neg t$



Precedence

- Should $\neg q \rightarrow \neg p$ be interpreted as
  - $(\neg q) \rightarrow (\neg p)$, or as
  - $\neg(q \rightarrow (\neg p))$?
- Precedence gives rules for implicit parentheses

Op	Prec
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Semantics of Propositional Formulas

- A formula defines a function from truth assignments to truth values
  - A truth assignment gives a truth value for each variable

Implications Related to $p \to q$

- *Contrapositive*: $\neg q \to \neg p$
- *Converse*: $q \to p$
- *Inverse*: $\neg p \to \neg q$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

Consistency

- Intuitive requirement: specifications should not contain conflicting requirements
- Definition: A collection of propositional formulas is *consistent* if there is a truth assignment that makes each formula true

Example Inconsistent Spec

- Specification:
  - Fred can access the wireless network only if Fred has paid his tuition ($a \to t$)
  - Fred can access the wireless network (a)
  - Fred has not paid his tuition ($\neg t$)

a	t	$a \to t$	$\neg t$	$(a \to t) \wedge a \wedge \neg t$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

- $(a \to t) \wedge a \wedge \neg t$ is a *contradiction*

A More Concise Truth Table

((*a*	→	*t*)	∧	*a*)	∧	¬	*t*
T		T		T			T
T		F		T			F
F		T		F			T
F		F		F			F

A More Concise Truth Table

$((a$	$\rightarrow$	$t)$	$\wedge$	$a)$	$\wedge$	$\neg$	t
T	T	T		T		F	T
T	F	F		T		T	F
F	T	T		F		F	T
F	T	F		F		T	F

A More Concise Truth Table

$((a$	$\rightarrow$	$t)$	$\wedge$	$a)$	$\wedge$	$\neg$	t
T	T	T	T	T		F	T
T	F	F	F	T		T	F
F	T	T	F	F		F	T
F	T	F	F	F		T	F

A More Concise Truth Table

$((a$	$\rightarrow$	$t)$	$\wedge$	$a)$	$\wedge$	$\neg$	t
T	T	T	T	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	F	F	F	F	T
F	T	F	F	F	F	T	F

Tautologies and Contradictions

- A compound proposition that is true for all truth assignments is a *tautology*
  - *E.g.*, $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$
- One that is false for all assignments is a *contradiction*
- Given propositional formulas *p* and *q*, if the biconditional $p \leftrightarrow q$ is a tautology, then *p* and *q* are *logically equivalent*
  - In this case we write $p \equiv q$
    - e.g., $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$
  - Note that ≡ is not a logical connective (*i.e.,* a logical operator), so $p \equiv q$ is not a compound proposition
  - ≡ is part of the *meta-language* we use for discussing formulas in the *object language* of propositional calculus

Some Important Logical Equivalences

$\phi \wedge T \equiv \phi$ $\phi \vee F \equiv \phi$	Identity laws
$\phi \vee T \equiv T$ $\phi \wedge F \equiv F$	Domination laws
$\phi \vee \phi \equiv \phi$ $\phi \wedge \phi \equiv \phi$	Idempotency laws
$\neg(\neg \phi) \equiv \phi$	Double nagation law
$\phi \vee \psi \equiv \psi \vee \phi$ $\phi \wedge \psi \equiv \psi \wedge \phi$	Commuitivity laws

More Equivalences

$(\phi \vee \psi) \vee \theta \equiv \phi \vee (\psi \vee \theta)$ $(\phi \wedge \psi) \wedge \theta \equiv \psi \wedge (\phi \wedge \theta)$	Associativity laws
$\phi \vee (\psi \wedge \theta) \equiv (\phi \vee \psi) \wedge (\phi \vee \theta)$ $\phi \wedge (\psi \vee \theta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \theta)$	Distributivity laws
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$	De Morgan's laws
$\phi \vee (\phi \wedge \psi) \equiv \phi$ $\phi \wedge (\phi \vee \psi) \equiv \phi$	Absorption laws
$\phi \vee \neg\phi \equiv \mathrm{T}$ $\phi \wedge \neg\phi \equiv \mathrm{F}$	Negation laws