

Discrete Mathematical Structures CS 3233 Lecture Three

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Business

- Questions???
- Read Section 1.4 by Tuesday
- Recall from last time: **Homework due Tuesday September 5**
 - Section 1.1: 6d, 6e, 6g, 10, 18a, 18c, 18e, 26, 28
 - Section 1.2: 2, 6, 10, 12
 - Turn in at beginning of class
 - No late assignments accepted

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Some Important Logical Equivalences

$\phi \wedge \mathbf{T} \equiv \phi$ $\phi \vee \mathbf{F} \equiv \phi$	Identity laws
$\phi \vee \mathbf{T} \equiv \mathbf{T}$ $\phi \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$\phi \vee \phi \equiv \phi$ $\phi \wedge \phi \equiv \phi$	Idempotency laws
$\neg(\neg \phi) \equiv \phi$	Double Negation law
$\phi \vee \psi \equiv \psi \vee \phi$ $\phi \wedge \psi \equiv \psi \wedge \phi$	Commutivity laws

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More Equivalences

$(\phi \vee \psi) \vee \theta \equiv \phi \vee (\psi \vee \theta)$ $(\phi \wedge \psi) \wedge \theta \equiv \phi \wedge (\psi \wedge \theta)$	Associativity laws
$\phi \vee (\psi \wedge \theta) \equiv (\phi \vee \psi) \wedge (\phi \vee \theta)$ $\phi \wedge (\psi \vee \theta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \theta)$	Distributivity laws
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$	De Morgan's laws
$\phi \vee (\phi \wedge \psi) \equiv \phi$ $\phi \wedge (\phi \vee \psi) \equiv \phi$	Absorption laws
$\phi \vee \neg\phi \equiv \mathbf{T}$ $\phi \wedge \neg\phi \equiv \mathbf{F}$	Negation laws

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An Equivalence for Implication

Theorem:

Given any propositional formulas ϕ and ψ ,
 $\phi \rightarrow \psi \equiv \neg\phi \vee \psi$

Proof:

ϕ	ψ	$\phi \rightarrow \psi$	$\neg\phi \vee \psi$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

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In-class Exercises

- Section 1.2: 9, 11
- Which of the following are propositional formulas?
 - pq
 - $((p))$
 - $q \vee \neg \neg \neg r$
 - $p \vee \rightarrow q$

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Syntax of Propositional Formulas (Review)

- Definition of *propositional formula*:
 - A propositional variable p is a propositional formula
 - The constants **T** and **F** are propositional formulas
 - If ϕ and ψ are propositional formulas, then the following are also propositional formulas:
 - (ϕ)
 - $\neg\phi$
 - $\phi \wedge \psi$
 - $\phi \vee \psi$
 - $\phi \rightarrow \psi$
 - $\phi \leftrightarrow \psi$
 - $\phi \oplus \psi$

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Section 1.3: Predicates

- Propositional functions
- Examples
 - $p(x) \equiv x > 3$
 - $q(y) \equiv y$ has paid his tuition
 - $r(z) \equiv z$ has wireless access on campus
- $p(4)$ has a truth value
- What does $r(x) \rightarrow q(x)$ mean?

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Predicates, Continued

- Each propositional function takes a fixed number of arguments
 - $\text{older}(x,y) \equiv x$ is older than y
 - Here “older” is being used as a propositional function
- A propositional function p and a predicate p are the same thing
- A statement of the form $p(x_1, x_2, x_n)$ is a proposition

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Formulas Involving Predicates

- Let $\phi(y) \equiv q(y) \rightarrow r(y)$
 - We say ϕ is a *formula in y* (not in text)
- Then $\phi(\text{Fred}) \equiv q(\text{Fred}) \rightarrow r(\text{Fred})$
 - This is an example of substitution of variables by values

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Universe of Discourse and Quantifiers

- The *universe of discourse* or *domain* is the set of all possible values for variables
- We can refer to values in the universe either by using constant symbols (like “Fred”) or by using quantifiers
- There are two quantifiers in standard predicate calculus: *for all* (\forall) and *there exists* (\exists)
- There are called the universal quantifier and the existential quantifier, respectively

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Universal Quantifiers

- The universal quantification of $p(x)$ is the following proposition:
 - “ $p(x)$ is true for all values of x in the universe of discourse”
 - Written $\forall x p(x)$ or $\forall x.p(x)$
- Similarly, if $\phi(x)$ is a formula in x , $\forall x.\phi(x)$ means the formula holds for all elements of the universe
 - What does $\forall x.(r(x) \rightarrow q(x))$ mean?
 - How is it different from $(\forall x.r(x)) \rightarrow (\forall x.q(x))$?

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Scope of Quantifiers

- $(\exists x.x > 3) \wedge (\exists x.x < 1)$
- The scope of a quantifier is the formula following the dot

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Universal Quantifiers

- Note that $\forall x.p(x) \equiv \forall y.p(y)$
- If the universe of discourse is $\{0, 1, 2\}$, then $\forall x.p(x) \equiv p(0) \wedge p(1) \wedge p(2)$
- Can you always rewrite $\forall x.p(x)$ this way?
 - What if the universe of discourse is infinite?
 - Logical statements are finite objects

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Existential Quantifiers

- The existential quantification of $p(x)$ is the proposition
 - “there exists an element x in the universe of discourse such that $p(x)$ is true”
 - Written $\exists x p(x)$ or $\exists x.p(x)$
 - What does $\exists x.(r(x) \wedge \neg q(x))$ mean?
- If the universe of discourse is $\{0, 1, 2\}$, then
 - $\exists x.p(x) \equiv p(0) \vee p(1) \vee p(2)$

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Negations of Quantified Formulas

- $\neg \exists x.p(x) \equiv \forall x.\neg p(x)$
- $\neg \forall x.p(x) \equiv \exists x.\neg p(x)$

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Predicate Formulas as Specifications

- For the domain of integers, is $\forall x.(x > 3)$ true or false?
 - How about $\exists x.(x > 3)$?
 - Note: “>” is the predicate symbol here
 - $X > 3$ is another way of writing $>(x, 3)$
- What does $\forall x \forall y.(x + y = y + x)$ mean?
 - Is it true?
 - What is the predicate here?
 - In logic, “+” is called a *function symbol* (term not introduced in the text)

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Open and Closed Formulas

- Not in text
- A formula is *closed* if all variables are in the scope of some quantifier
- Otherwise, the formula is *open*
- $p(x) \equiv \forall y.x \leq y$ is open

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