Discrete Mathematical Structures
CS 3233 Lecture Nine

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Business

- Read Sections 2.4 for Thursday
- Recall **Homework Four**
  Due Tuesday September 26:
  - Section 2.3: 4, 10, 14, 16, 38
  - Section 2.4: 4, 8
- Questions???

Examples

- Successor function
  - $s : N \to N$
  - $s(n) = n+1$
- Floor
  - $\text{floor} : R \to Z$
  - floor(x) = greatest integer less than x
- Square root
  - $\sqrt{\ } : R \to C$
  - The square root of a real is a complex number
- Truth assignment
  - $\sigma : \mathcal{V} \to \{\text{true}, \text{false}\}$

Injections and Surjections

- $f : A \to B$ is *one-to-one*, or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$
  - $f : A \xrightarrow{1-1} B$
  - $\forall x \forall y (f(x) = f(y) \to x = y)$
  - Equivalently, $\forall x \forall y (x \neq y \to f(x) \neq f(y))$
  - f is called an *injection*
- $f : A \to B$ is *onto,* or *surjective*, if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$
  - f is *a function of* A *onto* B
  - $f : A \xrightarrow{\text{onto}} B$
  - $\forall y \exists x (f(x) = y)$
  - f is called a *surjection*

More Examples

- Square
  - $f : Z \rightarrow Z$
  - $f(x) = x^2$
  - Neither one-to-one nor onto
- Double
  - twice : $Z \rightarrow Z$
  - twice(x) = 2x
  - One-to-one, but not onto
- Absolute value
  - $|\cdot| : Z \rightarrow N$
  - $|x| = \begin{cases} x \text{ if } 0 \leq x; \\ -x, \text{ otherwise} \end{cases}$
  - Onto, but not one-to-one

Still More Examples

- Integer successor
  - zs : $Z \rightarrow Z$
  - zs(n) = n+1
  - Both one-to-one and onto
- Mapping of 32-bit words to Z
  - Injective, but not surjective
- Successor in Z_3 = {0, 1, 2}
  - s(0) = 1
  - s(1) = 2
  - s(2) = 0
  - Injective and surjective

Bijections

- If $f : A \rightarrow B$ is injective and surjective, it is *bijective*
  - f is called a *one-to-one correspondence,* or a *bijection*

Compositions

- Definition
  - Given $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of f and g, $f \circ g : A \rightarrow C$, is defined by $f \circ g(a) = f(g(a))$
  - Note that no special properties of f and g are required for $f \circ g$ to be defined.
    - For instance, f and g need not be injective, surjective, or bijective
    - However, if f and g have special properties, it often follows that $f \circ g$ special properties as well
    - Study hint: think through these relationships

Function Inverses

- Theorem: Given f : A $\xrightarrow[\text{onto}]{\text{1-1}}$ B, every b ∈ B has a unique pre-image a ∈ A
- This justifies the following definition: Given f : A $\xrightarrow[\text{onto}]{\text{1-1}}$ B
  - f^{-1} : B → A is the *inverse of* f
  - $f^{-1}(b) = a$ iff $f(a) = b$

Sequences

- Definition
  - A *sequence* is a function from $\mathbb{N}$ or $\mathbb{Z}^+$ to a given set S
    - We use a_n to denote the image of n
    - We use $\{a_n\}$ to denote the whole sequence
      - Less formally, we sometimes denote it by $\{a_0, a_1, a_2, a_3, \ldots\}$
  - If the function is onto we say $\{a_n\}$ enumerates S

Bijections and cardinality

- Sets A and B are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them
  - Notice that this defines a binary relation over sets
  - Contains the ordered pairs of sets between which there exists a one-to-one correspondence
- Pretty clear for finite sets

Equinumerous Infinite Sets

- N and Z have the same cardinality
  - f : N → Z
  - $f(n) = \begin{cases} 0, & \text{if } n = 0 \\ n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd} \end{cases}$
  - Note, f is a bijection
    - Use case analysis to show f is injective based on whether x and y are odd or even
- Show N and Z^+ are equinumerous

Countability

- Definition
  - A set A is *countable* if it is finite or it is equinumerous to Z^+
  - Otherwise, A is *uncountable*
- Theorem: Q^+, the set of positive rationals, is countable

Countability and Enumeration

- Theorem: S is countable if and only if there exists a sequence that enumerates S
- Proof
  - only if: If there is a bijection between N and S, it is a sequence that enumerates S
  - if: Given a sequence that enumerates S, either S is finite or dropping repeated values from the sequence yields a bijection between N and S

Uncountability of the Reals

- Theorem
  - R, the set of real numbers, is uncountable
- Proof
  - Uses Georg Cantor's *diagonalization argument*
  - Outline
    - Assume for contradiction that there is a one-to-one correspondence, f, between N and the real interval [0,1]
    - Use f to construct a real in [0,1] that has no pre-image under f
    - Idea: for each decimal place, n, in the representation of the constructed value, choose a decimal different from the n^{th} place of f(n)
    - The fact that the constructed value differs from each value assumed by f shows that f is not onto, giving the desired contradiction

Conclusion

- So the cardinalities of N, Z, and Q are all the same
- But the cardinality of R is different