Discrete Mathematical Structures
CS 3233 Lecture Ten

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Business

- No homework this week
- Practice exam will be posted by Thursday
- Questions???

Quick Review

- Given two sets, A and B, A *binary relation* r between A and B is a subset of AxB
- What is required for r to be a function?
- If r is a function, then
  – What is required for it to be injective?
  – What is required for it to be surjective?

Countability

- Definition
  – A set A is *countable* if it is finite or it is equinumerous to Z^+
  – Otherwise, A is *uncountable*
- Theorem: Q^+, the set of positive rationals, is countable

Countability and Enumeration

- Theorem: S is countable if and only if there exists a sequence that enumerates S
- Proof
  – Only if: If there is a bijection between N and S, it is a sequence that enumerates S
  – If: Given a sequence that enumerates S, either S is finite or dropping repeated values from the sequence yields a bijection between N and S

Countability Exercise 1

- Theorem: If A is an uncountable set and B is a countable set, A – B is uncountable
- Proof
  – Suppose for contradiction that A – B is countable
  – This means that there is a sequence that enumerates all elements of A – B
  – We can now construct a sequence that enumerates A
    - It alternates between the sequence that enumerates A – B and the sequence that enumerates B
    - This contradicts the assumption that A is uncountable
  – It follows that the assumption A – B is countable is false, hence A – B is uncountable

Countability Exercise 2

- Theorem: If A is an uncountable set and A is a subset of B, then B is uncountable
- Proof:
  – Suppose B is countable
  – This means there is a sequence that enumerates B
  – A sequence that enumerates A can be constructed by dropping the elements of B – A, yielding the desired contradiction

Uncountability of the Reals

- Theorem
  – R, the set of real numbers, is uncountable
- Proof
  – Uses Georg Cantor's *diagonalization argument*
  – Outline
    - Assume for contradiction that there is a one-to-one correspondence, f, between N and the real interval [0,1]
    - Use f to construct a real in [0,1] that has no pre-image under f
    - Idea: for each decimal place, n, in the representation of the constructed value, choose a decimal different from the n^{th} place of f(n)
    - The fact that the constructed value differs from each value assumed by f shows that f is not onto, giving the desired contradiction