Discrete Mathematical Structures
CS 3233 Lecture Five

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Business

- Read Sections 1.6 and 1.7 by Tuesday
  (You can stop on p.97 before "Tilings".)
- **Recall: Homework Two:** due Tuesday September 12
  - Section 1.3: 10d, 10e, 14, 24c, 24d, 32a, 32b, 44
  - Section 1.4: 2a, 2c, 4, 10, 24, 46
  - Section 1.5: 4, 8
- Questions???
- Return Homework One
  - Will be discussed in recitation
- Practice quiz

Practice Quiz: In Which Numeric Domains Does each of the Following Hold?

- Domains
  - Z – The integers
  - N – The natural numbers (non-negative integers)
  - R^+ - The positive reals
  - $R^+ \cup \{0\}$ – The non-negative reals
- Formulas
  - $\forall x.\exists y.y<x$
  - $\exists x.\forall y.x \leq y$
  - $\forall x.\forall z.(x<z \rightarrow \exists y.(x<y) \wedge (y<z))$

Free and Bound Variables

- A variable occurrence in a formula is called *bound* if it lies in the scope of a quantification of that variable:
  - All bound: $\forall x.\forall z.(x<z \rightarrow \exists y.(x<y) \wedge (y<z))$
  - Free y: $\forall x.\forall z.(x<z \rightarrow (x<y) \wedge (y<z))$
  - Free x and bound x: $x<7 \wedge \forall x.(p(x) \rightarrow x>1)$
- Otherwise a variable occurrence is called *free*

Open and Closed Formulas

- A formula is *closed* if all variable occurrences are bound
  - Otherwise, it is *open*
  - *E.g.*, $\Phi(x) = \forall y.x \leq y$ is open
- When denoting a formula by a meta-variable like Φ, we list the variables that occur free in the formula after the variable: $\Phi(x)$
  - Can refer to an instance of $\Phi(x)$ by providing values for free variables: $\Phi(5) = \forall y.5 \leq y$
  - (In what numeric domains does $\Phi(0)$ hold?)

Theorems and Proofs

- A *theorem* is a statement (such as a formula) that can be shown to be true in all cases (a tautology)
- A proof is a demonstration that a statement is a theorem
- Example methods of proof
  - Construction of truth tables
  - Use of equivalences
    - By using these alone, can prove only logical equivalences
  - More general rules of inference

Important Related Terminology

- *Result*: often used to mean a theorem
- *Proposition*: a simple theorem, often presented without proof
- *Lemma*: a theorem whose main utility lies in helping to prove other, more interesting theorems
- *Corollary*: a theorem that follows easily from another more general theorem
- *Conjecture*: a statement that you suspect is true but that you do not yet have a proof for

Rules of Inference for Propositional Logic

- A general, systematic method of proving formulas
- See Table 1 p.66
  - Known equivalences can also be used in proofs
- Use rules of inference to show
  - These hypotheses:
    - If it does not rain or if it is not foggy, the sailing race will be held and the lifesaving demonstration will take place
    - If the race is held, the trophy will be awarded
    - The trophy was not awarded
  - Imply this conclusion:
    - It rained

Rules of Inference for Universally Quantified Statements

- Universal instantiation

$$\frac{\forall x.\ p(x)}{p(c)}$$

  - For any c
- Universal generalization

$$\frac{p(c)}{\forall x.\ p(x)}$$

  - Must show p(c) for arbitrary c

Rules of Inference for Existentially Quantified Statements

- Existential instantiation

$$\frac{\exists x.\ p(x)}{p(c)}$$

  - c must be a new name (constant) that does not appear earlier in the proof
- Existential generalization

$$\frac{p(c)}{\exists x.\ p(x)}$$

  - c can be any name

Universal Modus Ponens

- Combines propositional modus ponens with universal instantiation:

$$\frac{\begin{array}{l}\forall x.\Phi(x)\rightarrow\Psi(x)\\ \Phi(c)\end{array}}{\Psi(c)}$$

Using Rules of Inference for Quantified Statements

- Problems 14 and 15, p.73
- We won't always be so meticulous about identifying exactly which rules of inference are being used
  - But it's important to understand the fundamentals of how proofs can be constructed