Solutions: Mock Exam for Midterm II
Discrete Mathematical Structures CS3233

November, 2006

1. Define $f(n) = \mathcal{O}(g(n))$, $f(n) = \Omega(g(n))$, and $f(n) = \Theta(g(n))$.
   Solution: Refer to Text and Lecture Notes.

2. Prove or disprove: $(n^2 + n^3)/2 = \Theta(n^3)$.
   Solution: We must show that $(n^2 + n^3)/2 = \mathcal{O}(n^3)$ and $(n^2 + n^3)/2 = \Omega(n^3)$. Taking $n > 1$, we have $(n^2 + n^3)/2 \leq n^3$ and $(n^2 + n^3)/2 \geq n^3/2$, which demonstrate the respective conditions.

3. Prove or disprove: $n^2 \log n + n^2 = \Theta(n^2)$. **Solution:** We show that $n^2 \log n + n^2$ is not $\mathcal{O}(n^2)$. Assume for contradiction that it is and that there exist k and C such for all $n \geq k$, $n^2 \log n + n^2 \leq Cn^2$. Now consider any n such that $n > 2^C$ and $n \geq k$. For such n we have $\log n > C$ and hence $n^2 \log n > Cn^2$, which entails $n^2 \log n + n^2 > Cn^2$, giving us the desired contradiction.

4. What is the best big-O estimate of the number of comparisons that are performed by an algorithm that takes a list of n integers and finds the least of the first 100 values? Justify your answer.
   Solution: $\mathcal{O}(1)$. No matter how large n may be, the alorithm looks at a constant number of elements (at most 100). The remaining elements need not be inspected or manipulated in any way.

5. What is the worst-case complexity of finding the least value in a list of n integers? Select the one best answer from the following list: $\mathcal{O}(1), \mathcal{O}(\log n), \mathcal{O}(n), \mathcal{O}(n \log n), \mathcal{O}(n^2), \mathcal{O}(n^3), \mathcal{O}(2^n)$?
   Solution: $\mathcal{O}(n)$. It is not possible to find the least value without looking at all the values: if an algorithm skipped any value and it happened to be the least one, the algorithm would be wrong.

6. What is the worst-case time complexity of using binary search to find determine whether a given value is in a given sorted list of integers? Assume that the time required to obtain the sorted list as input is negligible, as if, say, it were already available in memory. Select the one best answer from the following list: $\mathcal{O}(1), \mathcal{O}(\log n), \mathcal{O}(n), \mathcal{O}(n \log n), \mathcal{O}(n^2), \mathcal{O}(n^3), \mathcal{O}(2^n)$?
   Solution:$\mathcal{O}(\log n)$

7. Use mathematical induction to prove that $n^3 - n$ is divisible by 3 for all natural numbers n
   Solution: For the base case, we need only observe that when $n = 0$, the expression is also 0 and hence divisible by 3. $(3 \times 0 = 0)$
   For the inductive step, we assume that $n^3 - n$ is divisible by 3 and consider $(n+1)^3 - (n+1) = n^3 + 3n^2 + 2n = (n^3 - n) + 3(n^2 + n)$. The fact that the latter quantity is divisible by 3 now follows because $(n^3 - n)$ is divisible by 3 by the induction assumption and $3(n^2 + n)$ is obviously divisible by 3.

8. Use induction to show that $P(n) \equiv \sum_{1 \leq i \leq n} (-1)^{i-1} i^2 = (-1)^{n-1} n(n+1)/2$ holds for all positive integers n. **Solution:** In the base case, when $n = 1$, both expressions take on the value 1.
   In the inductive step, we proceed as follows:

$$\begin{aligned}
&\sum_{i=1}^{n+1} (-1)^{i-1} i^2 \\
&= \textstyle\sum_{i=1}^{n} (-1)^{i-1} i^2 + (-1)^n (n+1)^2 \\
&= \frac{(-1)^{n-1} n(n+1)}{2} + \frac{2(-1)^n (n^2+2n+1)}{2} \qquad \text{by induction assumption} \\
&= \frac{(-1)^{n-1}(n^2+n) + (-1)^n (2n^2+4n+2)}{2} \\
&= \frac{(-1)^n (n^2+3n+2)}{2} \\
&= \frac{(-1)^n (n+1)(n+2)}{2}
\end{aligned}$$

9. Is the set of negative integers well ordered? Why or why not?
   Solution: It is not because any subset containing an infinite number of negative integers has no least element.

10. Is the set of integers greater than 100 well ordered? Why or why not?
    Solution: It is because any subset of this set is also a subset of N, so it has a least element, since N is well ordered.

11. Determine whether the following are valid recursive definitions of a function $f : N \to Z$:

    (a) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n-2)$ for $n > 1$

    (b) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n)$ for $n > 1$

    (c) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n+1) + f(n+2)$ for $n > 1$

    Solution: valid, invalid, invalid

12. This question may or may not be relevent to midterm II, depending on how far we get in lecture.

    Let the set of positive propositional formulas Pos be recursively defined as follows:
    Basis: $p \in Pos$ for all propositional variables p.
    Recursive step: if $\varphi \in Pos$ and $\psi \in Pos$, then $\varphi \wedge \psi \in Pos$, $\varphi \vee \psi \in Pos$, and $(\varphi) \in Pos$.

    Given truth assignments σ_1 and σ_2, we write $\sigma_1 \preceq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)

    Prove the following by structural induction: Consider any $\varphi \in Pos$ and and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preceq \sigma_2$. If σ_1 satisfies φ, then so does σ_2.

    Solution:
    Basis: If φ is a propositional variable and σ_1 satisfies φ, then so does σ_2 by definition of $\sigma_1 \preceq \sigma_2$.

    inductive step: Suppose for the induction hypothesis that if σ_1 satisfies φ then σ_2 satisfies φ and that if σ_1 satisfies ψ then σ_2 satisfies ψ. We show that if σ_1 satisfies $\varphi \vee \psi$ then σ_2 satisfies $\varphi \vee \psi$. A complete answer to this problem also requires one to prove that if σ_1 satisfies $\varphi \wedge \psi$ then σ_2 satisfies $\varphi \wedge \psi$ and that if σ_1 satisfies (φ) then σ_2 satisfies (φ). However, we omit these cases here, as they are similar.

    We assume the atecedent and show the consequent. Since σ_1 satisfies $\varphi \vee \psi$, it must be that σ_1 satisfies φ or σ_1 satisfies ψ. In the former case it follows by using the induction assumption that σ_2 satisfies φ, from which it follows that σ_2 satisfies $\varphi \vee \psi$, as desired. We obtain the desired conclusion similarly in the case that σ_1 satisfies ψ.