Mock Exam for Midterm I
Discrete Mathematical Structures CS3233

The actual exam will be closed-book and will include true/false and multiple-choice questions. In the actual exam, you will write your answers on the same paper on which the questions are printed. This collection of problems is intended to be representative of the material that is important on the exam. It is longer than the actual exam, but the actual exam is a little long, too. The actual exam is also rather hard. It will be graded on a curve.

In addition to the problems given here, the problems from in-class practice quizes are also representative of the questions on the exam.

1. Define each of the following: tautology, consistent, inconsistent, valid, contradiction, logical equivalence, contrapositive, converse.
   Solution: Refer to Sections 1.2, 1.5 and Lectures 1.

2. What are some universes of discourse in which the following are true?

   (a) $\forall x \exists y (y > x)$
   Solution: True: N, Z^+, Z, R, R^-, Q^-. False: Z^-

   (b) $\exists x \forall y (x \neq y \rightarrow x < y)$
   Solution: True: N, Z^+. False: Z^-, Z, R, R^+, Q^+

   (c) $\forall x \forall y ((x < y) \rightarrow (\exists z (x < z \wedge z < y)))$
   Solution: True: R, Q. False: N, Z

3. Prove that the tautology corresponding to the *Resolution* rule of inference is true by using truth tables. (See Table 1 of Section 1.5, p.66.)
   Solution:

[(p	∨	q)	∧	(¬	p	∨	r)]	→	(q	∨	r)
T	T	T	T	F	T	T	T	T	T	T	T
T	T	T	F	F	T	F	F	T	T	T	F
T	T	F	T	F	T	T	T	T	F	T	T
T	T	F	F	F	T	F	F	T	F	F	F
F	T	T	T	T	F	T	T	T	T	T	T
F	T	T	T	T	F	T	F	T	T	T	F
F	F	F	F	T	F	T	T	T	F	T	T
F	F	F	F	T	F	T	F	T	F	F	F

4. Write a formal, logical proof similar to the ones in Examples 6 and 7 of Section 1.5 proving q from the assumptions p and $p \vee r \rightarrow q$.
   Solution:

Step	**Justification**
p	Assumption
$p \vee r$	Addition
$p \vee r \rightarrow q$	Assumption
q	modus ponens

5. Write a formula that says there is no least number. Is the formula true when the universe of discourse is Z (the set of integers)? When it is R^+ (the positive reals)? When it is N (the set of natural numbers)?
   Solution: $\forall x \exists y (y < x)$
   Z: yes. R^+: yes. N: no.

6. Write a formula that is true if and only if the universe of discourse is *dense*, meaning that between any two distinct numbers there is a third distinct number.
Solution: $\forall x \forall z(x < z \rightarrow \exists y(x < y \wedge y < z))$

7. Write formulas that holds if and only if $S \subseteq A \times B$ is:

   (a) A function.
   Solution: $\forall (a_1, b_1) \in S \, \forall (a_2, b_2) \in S (a_1 = a_2 \rightarrow b_1 = b_2)$
   Note that this formula does not require S to be a total function, which is the kind of function we normally deal with. The formula below would have to be included for that.

   (b) One-to-one.
   Solution: $\forall (a_1, b_1) \in S \, \forall (a_2, b_2) \in S (b_1 = b_2 \rightarrow a_1 = a_2)$

   (c) Total.
   Solution: $\forall a \in A \, \exists b \in B ((a, b) \in S)$

   (d) Onto.
   Solution: $\forall b \in B \, \exists a \in A ((a, b) \in S)$

8. What is the powerset of $S = \{0, 1\}$?
Solution: $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$

9. What is the definition of a *rational* number?
Solution: A rational number is the quotent of two integers.

10. Given a finite set S of size n, how many elements are there in $\wp(S)$, the powerset of S?
Solution: 2^n

11. Use the definitions of union, intersection, and set complement, as well as De Morgan's First Law of *logical* equivalence (namely, $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$) to prove De Morgan's first set identity (namely, $\overline{A \cup B} = \overline{A} \cap \overline{B}$).
Solution:

$$
\begin{array}{lll}
\overline{A \cup B} & & \\
\equiv & \{x | x \in \overline{A \cup B}\} & \text{Definition of set builder notation} \\
\equiv & \{x | \neg(x \in A \cup B)\} & \text{Definition set compliment} \\
\equiv & \{x | \neg((x \in A) \vee (x \in B))\} & \text{Definition set union} \\
\equiv & \{x | (\neg(x \in A)) \wedge (\neg(x \in B)))\} & \text{De Morgan's First Law} \\
\equiv & \{x | (x \in \overline{A}) \wedge (x \in \overline{B}))\} & \text{Definition set compliment} \\
\equiv & \{x | (x \in \overline{A} \cap \overline{B})\} & \text{Definition set intersection} \\
\equiv & \overline{A} \cap \overline{B} & \text{Definition of set builder notation}
\end{array}
$$

12. Prove there is a one-to-one correspondence between the integers and the even integers.
Solution: The function $f(x) = 2x$ is a one-to-one correspondence. It is injective because $2u = 2v \rightarrow u = v$ and it is surjective because every even integer is $2x$ for some integer x.

13. Given sets A, B, and C, prove that if there is a one-to-one between A and B, and there exists a one-to-one correspondence between B and C, then there exists a one-to-one correspondence between A and C.
Solution: By assumption, there are bijective functions $f : A \rightarrow B$ and $g : B \rightarrow C$. We prove that $g \circ f$ is bijective. For injectivity, consider any distinct $a_1, a_2 \in A$. Because f is injective, $f(a_1)$ and $f(a_2)$ are distinct elements of B. So, since g is injective, $g(f(a_1))$ and $g(f(a_2))$ are distinct. Since $g \circ f(a_1) = g(f(a_1))$ and $g \circ f(a_2) = g(f(a_2))$, $g \circ f$ is injective. For surjectivity, we show that for all $c \in C$ there is an $a \in A$ such that $g \circ f(a) = g(f(a)) = c$. Since g is surjective, there exists a $b \in B$

such that $g(b) = c$. Fix such a b. Now since f is surjective, there exists an $a \in A$ such that $f(a) = b$. Fixing such an a, we have the desired result.

Note that when I say "fix such a b," what I mean is that in the rest of the argument, when I refer to b, I mean one particular, arbitrarily chosen value having the property that $g(b) = c$. This is a subtile technical point that I won't be testing you on at this stage, but that I needed to explain here.

14. What sequence is given by $s : N \to N$ defined by $s(x) = x + 1$?
Solution: $\{1, 2, 3, \ldots\}$.

15. Let R be the set of real numbers. Consider the following function: $rs : R \to R$ defined by $rs(x) = x+1$. Is rs a sequence?
Solution: No. This function's domain is the reals, while a sequence is a function whose domain is the naturals.

16. Prove that the Cartesian product of two countable sets is countable.
Solution: This is basically the same as the proof that Q^+ is countable. If you have sequences $\{a_i\}$ and $\{b_j\}$ enumerating countable A and B, then $A \times B$ is enumerated by the sequence that starts off $\{(a_0, b_0), (a_1, b_0), (a_0, b_1), (a_2, b_0), (a_1, b_1), (a_0, b_2), \ldots\}$. Basically, the sequence begins by enumerating (a_i, b_j) such that $i + j = 0$, then enumerates the pairs with $i + j = 1$, then $i + j = 2$, and so on.

17. Prove that the set of functions $N \to N$ is uncountable.
Solution: This is essentially the same at the proof that the set of reals is uncountable. We prove the result by contradiction. That is, we assume the set is countable and derive a contradiction. If the set is countable, then it can be enumerated by a sequence. To show that this is impossible, we consider any sequence of functions $\{f_i\}$ in $N \to N$ and show that there exists a function that is not contained in the sequence. One such function is $g(x) = f_x(x) + 1$. For each $n \in N$, g differs from f_n on some input, namely n. So g is different from every function in the sequence, as desired.

18. Prove that $\wp(N)$, the power set of the set of natural numbers, is uncountable.
Solution: Assume for contradiction that there is a sequence $\{S_i\}$ of subsets of N that enumerates all of $\wp(N)$. We construct a set T that is not in the sequence, thereby showing that no such sequence exists. Define T by $T = \{n \in N | n \notin S_n\}$. Now T differs from each element of the sequence at some point. In particular, T contains n if and only if S_n does not.