

Discrete Mathematical Structures CS 3233 Lecture Ten

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Introduction to Functions

- Given sets A and B , a *function from A to B* is an assignment of exactly one element of B to each element of A
- For $a \in A$, we write $f(a) = b$ if $b \in B$ is the unique element associated with a by f
- We write $f : A \rightarrow B$ to indicate that f is a function from A to B
 - $A \rightarrow B$ is called the *type* of f

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Function Graphs

- This is a different usage of "graph" than either of these:
 - The plot of $f(x)$ on the Cartesian plane
 - A set of nodes and a set of edges
- Definition
 - Given a function $f : A \rightarrow B$, the *graph* of f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } b = f(a)\}$
- Given two sets, A and B , A *binary relation* r between A and B is a subset of $A \times B$
- Formally speaking, a function is its graph
 - Thus, f is a subset of $A \times B$ in which each element of A occurs exactly once

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Terminology

- Let $f : A \rightarrow B$
 - We say f *maps* A to B
 - A is the *domain* of f
 - B is the *codomain* of f
 - If $f(a) = b$,
 - b is the *image* of a and
 - a is the *pre-image* of b
 - If $S \subseteq A$, the *image* of S is $\{f(s) \mid s \in S\}$ and is denoted by $f(S)$
 - $f(A) = \{f(a) \mid a \in A\}$ is called the *range* of f

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Examples

- Successor function
 - $s : \mathbb{N} \rightarrow \mathbb{N}$
 - $s(n) = n+1$
- Floor
 - $\text{floor} : \mathbb{R} \rightarrow \mathbb{Z}$
 - $\text{floor}(x)$ = greatest integer less than x
- Square root
 - $\sqrt{\cdot} : \mathbb{R} \rightarrow \mathbb{C}$
 - The square root of a real is a complex number
- Truth assignment
 - $\sigma : \mathcal{P}^2 \rightarrow \{\text{true}, \text{false}\}$

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Injections and Surjections

- $f : A \rightarrow B$ is *one-to-one*, or *injective*, if and only if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$
 - $f : A \xrightarrow{1:1} B$
 - $\forall x \forall y. f(x) = f(y) \rightarrow x = y$
 - Equivalently, $\forall x \forall y. x \neq y \rightarrow f(x) \neq f(y)$
 - f is called an *injection*
- $f : A \rightarrow B$ is *onto*, or *surjective*, if and only if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$
 - f is a *function of A onto B*
 - $f : A \xrightarrow{\text{onto}} B$
 - $\forall y \exists x (f(x) = y)$
 - f is called a *surjection*

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More Examples

- Injective and/or surjective?

- Square
 - $f : Z \rightarrow Z$
 - $f(x) = x^2$
- Double
 - $\text{twice} : Z \rightarrow Z$
 - $\text{twice}(x) = 2x$
- Absolute value
 - $| \cdot | : Z \rightarrow N$
 - $|x| = \begin{cases} x & \text{if } 0 \leq x; \\ -x, & \text{otherwise} \end{cases}$

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Still More Examples

- Injective and/or surjective?

- Integer successor
 - $zs : Z \rightarrow Z$
 - $zs(n) = n+1$
- Mapping of 32-bit words to Z
- Successor in $Z_3 = \{0, 1, 2\}$
 - $s(0) = 1$
 - $s(1) = 2$
 - $s(2) = 0$

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Bijections

- If $f : A \rightarrow B$ is injective and surjective, it is *bijective*
 - f is called a *one-to-one correspondence*, or a *bijection*

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Compositions

- Definition
 - Given $g : A \rightarrow B$ and $f : B \rightarrow C$, the composition of f and g , $f \circ g : A \rightarrow C$, is defined by $f \circ g(a) = f(g(a))$
 - Note that no special properties of f and g are required for $f \circ g$ to be defined.
 - For instance, f and g need not be injective, surjective, or bijective
 - However, if f and g have special properties, it often follows that $f \circ g$ special properties as well
 - Study hint: think through these relationships

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Function Inverses

- Theorem: Given $f : A \xrightarrow[1-1]{\text{onto}} B$, every $b \in B$ has a unique pre-image $a \in A$
- This justifies the following definition: Given

$$f : A \xrightarrow[1-1]{\text{onto}} B$$

- $f^{-1} : B \rightarrow A$ is the *inverse* of f
- $f^{-1}(b) = a$ iff $f(a) = b$

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Bijections and cardinality

- Sets A and B are *equinumerous* (meaning they have the same cardinality) iff there is a one-to-one correspondence between them
 - Notice that this defines a binary relation over sets
 - Contains the ordered pairs of sets between which there exists a one-to-one correspondence
- Pretty clear for finite sets

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Equinumerous Infinite Sets

- N and Z have the same cardinality
 - $f : \mathbb{N} \rightarrow \mathbb{Z}$
 - $f(n) = \begin{cases} 0, & \text{if } n = 0 \\ n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd} \end{cases}$
 - Note, f is a bijection
 - Use case analysis to show f is injective based on whether x and y are odd or even
- Show N and \mathbb{Z}^+ are equinumerous

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Countability

- Definition
 - A set A is *countable* if it is finite or it is equinumerous to \mathbb{Z}^+
 - Otherwise, A is *uncountable*
- Theorem: \mathbb{Q}^+ , the set of positive rationals, is countable

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Uncountability of the Reals

- Theorem
 - \mathbb{R} , the set of real numbers, is uncountable
- Proof
 - Uses Georg Cantor's *diagonalization argument*
 - Outline
 - Assume for contradiction that there is a one-to-one correspondence, f, between N and the real interval [0, 1]
 - Use f to construct a real in [0, 1] that has no preimage under f
 - Idea: for each decimal place, n, in the representation of the constructed value, choose a decimal different from the nth place of f(n)
 - The fact that the constructed value differs from each value assumed by f shows that f is not onto, giving the desired contradiction

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Conclusion

- So the cardinalities of N, Z, and Q are all the same
- But the cardinality of R is different

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