Discrete Mathematical Structures
CS 3233 Lecture Three

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Business

- Turn in Homework
- Questions???
- Kallen (our TA) will be available by appointment after class on Tuesday and Thursday
  - Arrange appointment with her at class time.
- Read Sections 1.2 and 1.3 by Tuesday
- **Homework due Thursday** September 6
  - Section 1.2: 2, 6, 10, 12
  - Section 1.3: 10d, 10e, 14, 24c, 24d, 32a, 32b, 44

Exercises

- Draw an expression tree for $p \rightarrow q \vee \neg r \wedge t$
- Which of the following are propositional formulas?
  - pq
  - ((p))
  - $q \vee \neg \neg \neg r$
  - $p \vee \rightarrow q$

Equivalence Revisited

- Given propositional formulas *p* and *q*, if the biconditional $p \leftrightarrow q$ is a tautology, then *p* and *q* are *logically equivalent*
  - In this case we write $p \equiv q$
    - e.g., $(p \rightarrow q) \equiv (\neg q \rightarrow \neg p)$
  - Note that ≡ is not a logical connective (*i.e.,* a logical operator), so $p \equiv q$ is not a compound proposition
  - ≡ is part of the *meta-language* we use for discussing formulas in the *object language* of propositional calculus

Consistency

- Intuitive requirement: specifications should not contain conflicting requirements
- Definition: A collection of propositional formulas is *consistent* if there is a truth assignment that makes each formula true

Some Important Logical Equivalences

$\phi \wedge T \equiv \phi$ $\phi \vee F \equiv \phi$	Identity laws
$\phi \vee T \equiv T$ $\phi \wedge F \equiv F$	Domination laws
$\phi \vee \phi \equiv \phi$ $\phi \wedge \phi \equiv \phi$	Idempotency laws
$\neg(\neg \phi) \equiv \phi$	Double Negation law
$\phi \vee \psi \equiv \psi \vee \phi$ $\phi \wedge \psi \equiv \psi \wedge \phi$	Commuitivity laws

More Equivalences

$(\phi \vee \psi) \vee \theta \equiv \phi \vee (\psi \vee \theta)$ $(\phi \wedge \psi) \wedge \theta \equiv \psi \wedge (\phi \wedge \theta)$	Associativity laws
$\phi \vee (\psi \wedge \theta) \equiv (\phi \vee \psi) \wedge (\phi \vee \theta)$ $\phi \wedge (\psi \vee \theta) \equiv (\phi \wedge \psi) \vee (\phi \wedge \theta)$	Distributivity laws
$\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$	De Morgan's laws
$\phi \vee (\phi \wedge \psi) \equiv \phi$ $\phi \wedge (\phi \vee \psi) \equiv \phi$	Absorption laws
$\phi \vee \neg\phi \equiv$ **T** $\phi \wedge \neg\phi \equiv$ **F**	Negation laws

An Equivalence for Implication

Theorem:
Given any propositional formulas ϕ and ψ,
$\phi \rightarrow \psi \equiv \neg\phi \vee \psi$

Proof:

ϕ	ψ	$\phi \rightarrow \psi$	$\neg\phi \vee \psi$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Exercise

- Use known equivalences to show the following:
  $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Section 1.3: Predicates

- Propositional functions
- Examples
  - p(x) ≡ x > 3
  - q(y) ≡ y has paid his tuition
  - r(z) ≡ z has wireless access on campus
- p(4) has a truth value
- What does r(x) → q(x) mean?

Predicates, Continued

- Each propositional function takes a fixed number of arguments
  - older(x,y) ≡ x is older than y
  - Here "older" is being used as a propositional function
- A propositional function p and a predicate p are the same thing
- A statement of the form $p(x_1, x_2, \ldots, x_n)$ is a proposition

Formulas Involving Predicates

- Let ϕ(y) ≡ q(y) → r(y)
  - We say ϕ is *a formula in* y (not in text)
- Then ϕ(Fred) ≡ q(Fred) → r(Fred)
  - This is an example of substitution of variables by values

Universe of Discourse and Quantifiers

- The *universe of discourse* or *domain* is the set of all possible values for variables
- We can refer to values in the universe either by using constant symbols (like "Fred") or by using quantifiers
- There are two quantifiers in standard predicate calculus: *for all* (∀) and *there exists* (∃)
- There are called the universal quantifier and the existential quantifier, respectively

Universal Quantifiers

- The universal quantification of p(x) is the following proposition:
  - "p(x) is true for all values of x in the universe of discourse"
  - Written ∀x p(x) or ∀x.p(x)
- Similarly, if ϕ(x) is a formula in x, ∀x.ϕ(x) means the formula holds for all elements of the universe
  - What does ∀x.(r(x) → q(x)) mean?
  - How is it different from (∀x.r(x)) → (∀x.q(x)) ?

Scope of Quantifiers

- (∃x.x>3) ∧ (∃x.x<1)
- The scope of a quantifier is the formula following the dot

Universal Quantifiers

- Note that ∀x.p(x) ≡ ∀y.p(y)
- If the universe of discourse is {0, 1, 2}, then ∀x.p(x) ≡ p(0) ∧ p(1) ∧ p(2)
- Can you always rewrite ∀x.p(x) this way?
  - What if the universe of discourse is infinite?
  - Logical statements are finite objects

Existential Quantifiers

- The existential quantification of p(x) is the proposition
  - "there exists an element x in the universe of discourse such that p(x) is true"
  - Written ∃x p(x) or ∃x.p(x)
  - What does ∃x.(r(x) ∧ ¬q(x)) mean?
- If the universe of discourse is {0,1,2}, then
  - ∃x.p(x) ≡ p(0) ∨ p(1) ∨ p(2)

Negations of Quantified Formulas

- ¬∃x.p(x) ≡ ∀x.¬p(x)
- ¬ ∀x.p(x) ≡ ∃x.¬p(x)

Predicate Formulas as Specifications

- For the domain of integers, is ∀x.(x>3) true or false?
  - How about ∃x.(x>3) ?
  - Note: “>” is the predicate symbol here
  - X>3 is another way of writing >(x,3)
- What does ∀x ∀y.(x+y = y+x) mean?
  - Is it true?
  - What is the predicate here?
  - In logic, “+” is called a *function symbol* (term not introduced in the text)

Open and Closed Formulas

- Not in text
- A formula is *closed* if all variables are in the scope of some quantifier
- Otherwise, the formula is *open*
- p(x) ≡ ∀y.x≤y is open

Exercises

- Section 1.2: 9, 11